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ON DISCRETE-IN-TIME PROBABILISTIC SCHEDULING PERIOD INVENTORY SYSTEM FOR DETERIORATING ITEMS WITH INSTANTANEOUS DEMAND

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ABSTRACT

A probabilistic scheduling period inventory system is considered for continuously deteriorating items in which the demand is assumed to occur instantaneously at the beginning of the scheduling period, shortages are not allowed, deterioration is assumed to be a constant fraction of the on hand inventory and time is treated as a discrete variable. Three inventory models are developed, viz. a model with no lead time, a model with a deterministic lead time and a model with a lead time equal to a multiple of the scheduling period. Examples are given to illustrate the derived results.

1. INTRODUCTION

In real life problems, time is usually treated as a discrete variable, e.g. 12 days, 3 weeks, 7 months, and so on. A result of, say, 13.4936 days makes little sense unless it is rounded off to mean either 13 or 14 days. Bearing this in mind, Dave[1] have introduced discrete-in-time inventory models for deteriorating items, most of which are deterministic in nature. Only a few discrete-in-time probabilistic inventory models for deteriorating items have been developed so far. A scheduling period inventory model has been developed by Dave and Jaiswal[4] which assumes zero lead time. Dave and Shah[5] have extended it for the case of a lead time which is exactly equal to the scheduling period of the system. A further generalisation has been provided by Dave[2] by considering a deterministic lead time which is independent of the scheduling period of the system. The common feature of these models is that they all are developed under the assumption of a uniform demand during the scheduling period. In many business organisations, the demands are fulfilled once a day, once a week, etc. This type of demand, known as the instantaneous demand pattern, is often encountered in practice. The demand can be instantaneous either at the beginning or at the end of the scheduling period depending upon the preference and the circumstances of the business.

In this paper, we consider a probabilistic scheduling period inventory system for deteriorating items with demand occurring instantaneously at the beginning of the scheduling period. The system does not allow shortages. Deterioration is a constant fraction of the on hand inventory at the beginning of a time unit. The problem is to find the optimal order-level of the system. To study the system, three inventory models are developed : (1) a model without lead time, (2) a model with a known lead time and (3) a model with a lead time equal to a multiple of the scheduling period. To develop the inventory models, time is treated as a discrete variable. In the absence of deterioration, these models are shown to be related to the inventory models of Dave[3]. Finally, each inventory model is illustrated with the help of a numerical example.

2.THE MODEL WITHOUT LEAD TIME

The following assumptions are made.

(1) The demand X during any scheduling period T is a random variable with a probability density function (p.d.f.) $f(x/T)$, $\{ -\infty < a(T) \leq X \leq b(T) < \infty \}$, with $R = \mu(T)/T$ as the average demand rate where

$$\mu(T) = E(X/T) = \int_{a(T)}^{b(T)} x.f(x/T)dx \qquad \text{------(1)}$$

is the mean demand during T. The demand X occurs instantaneously at the beginning of the scheduling period T immediately after the decision regarding the order-level S has been made. The p.d.f. $f(x/T)$ of the demand X during T is well-behaved so that all the expected values discussed hereafter exist. The distribution of demand is stationary over time.

(2) Replenishment rate is infinite. Replenishments are instantaneous. At the beginning of every scheduling period, which is of T time units long, a variable quantity is ordered so as to raise its initial inventory level to the order-level of S units.

(3) Shortages are not allowed.

(4) $Q(t)$ is the number of units in inventory at the beginning of the time unit t, $(t = 0,1,2,...,T)$. A constant fraction θ of the on hand inventory at the beginning of a time unit deteriorates during that time unit. There is no repair or replacement of the deteriorated inventory during the period under consideration.

(5) The relevant costs are the unit cost C, the inventory holding cost C_1 per unit per time unit and the replenishment cost C_3 per order. All the costs are known and constant during the period under consideration.

It is easy to see that the inventory position of the system during T is described by the difference equation

$$\Delta Q(t) + \theta Q(t) = 0 , \quad t = 0,1,2,...,T. \qquad \text{------(2)}$$

where $\Delta Q(t) = Q(t + 1) - Q(t)$.

Then, under the boundary condition $Q(0) = S - X$, the solution of (2) is

$$Q(t) = (S - X)(1 - \theta)^t, \quad t = 0,1,2,\ldots,T. \qquad \text{------(3)}$$

Since shortages are not allowed by the system, the order-level S must be sufficiently large so that even the maximum demand b(T) during the scheduling period T does not cause shortages. This suggests, from (3), that the order-level S must be given by

$$S = b(T) \qquad \text{------(4)}$$

Then, from (3), using (4), the expected number of units that deteriorate and the expected average number of units carried in inventory per time unit during the scheduling period T are

$$D(T) = S - \mu(T) - E\{Q(T)\} = \{b(T) - RT\}.A(T) \qquad \text{------(5)}$$

and

$$I_1(T) = \{1/(T + 1)\}\sum_{t=0}^{t=T} E\{Q(t)\}$$

$$= \{b(T) - RT\}.A(T + 1)/\theta(T + 1) \qquad \text{------(6)}$$

respectively, where

$$A(T) = 1 - (1 - \theta)^T \qquad \text{------(7)}$$

Since the number of replenishments per time unit is 1/T, from (5) and (6), the expected average total cost per time unit of the system during the period under consideration is

$$K(T) = CD(T)/T + C_1 I_1(T) + C_3/T$$

$$= \{b(T) - RT\}\{CA(T)/T + C_1 A(T+1)/\theta(T+1)\} + C_3/T \quad \text{---(8)}$$

At this point note that, when there is no deterioration, i.e. when $\theta = 0$, equation (8) reduces to

$$K(T) = C_1\{b(T) - RT\} + C_3/T \quad \text{------(9)}$$

which is the same as that given by Dave[3] for a similar model for non-deteriorating items.

For finding the optimal scheduling period T_o and thereby the optimal order-level S_o from (4), it is, now, necessary to know the explicit form of the function b(T), which gives rise to many particular cases. One such case is described below.

A Particular Case :-

$$\text{Let } b(T) = p\mu(T) = pRT \quad \text{-----(10)}$$

where $p \geqslant 1$ is a known constant. This means that $a(T) = r\mu(T) = rRT$ where $r \leqslant 1$ is also a known constant. Then, from (8), the expected average total cost of the system becomes

$$K(T) = (p-1)R\{CA(T) + C_1 TA(T+1)/\theta(T+1)\} + C_3/T \quad \text{-----(11)}$$

Since T must be a non-negative integer, the necessary condition for K(T) to be minimum at $T = T_o$ is

$$\Delta K(T_o - 1) \leqslant 0 \leqslant \Delta K(T_o) \quad \text{-----(12)}$$

where

$$\Delta K(T) = K(T + 1) - K(T) \qquad \text{-----(13)}$$

Using (11) and (13) in (12), the condition for optimality at $T = T_o$ is given by

$$M(T_o) \leqslant C_3/(p - 1)R \leqslant M(T_o + 1) \qquad \text{-----(14)}$$

where

$$M(T) = C\theta(T - 1)T(1 - \theta)^{(T-1)}$$

$$+ C_1(T - 1)\{1 - (1 - \theta T^2)(1 - \theta)^T\}/\theta(T + 1) \qquad \text{-----(15)}$$

Then, using (10) in (4), the optimal order-level is given by

$$S_o = pRT_o \qquad \text{-----(16)}$$

and the minimum total cost of the system can be obtained by substituting $T = T_o$ in (11).

Finally, in the absence of deterioration, $\theta = 0$ implies that $M(T) = C_1(T - 1)T$ from (15) and, then, the optimality condition (14) reduces to

$$(T_o - 1)T_o \leqslant C_3/C_1(p - 1)R \leqslant T_o(T_o + 1) \qquad \text{-----(17)}$$

which is the same as that derived by Dave[3] for a similar case for non-deteriorating items.

3.EXAMPLE 1

As an illustration to the above developed inventory model, consider a system with the parameter values given by C = \$40.00 per unit, C_1 = \$0.045 per unit/day, C_3 = \$200.00 per order and $\theta = 0.015$. The demand X during the scheduling period T follows the uniform distribution

$$f(x/T) = 1/50T \quad , \quad 0 \leq X \leq 50T$$
$$= 0 \quad , \quad \text{otherwise}$$

Then, $\mu(T) = 25T$ and $b(T) = 50T$ imply that $R = 25$ and $p = 2$. Also, note that, $C_3/(p - 1)R = 8$. For different values of T, the values of the functions M(T) and K(T) are given in table 1.

Table 1. Values of M(T) and K(T).

T	A(T)	M(T)	K(T)
1	0.015	0.000	216.125
2	0.030	1.270	132.200
3	0.044	3.753	113.985
4	0.059	7.393	113.380
5	0.073	12.137	118.418

Applying condition (14) to table 1, we find that the optimal value of the scheduling period is T_o = 4 days. The corresponding minimum total cost of the system is $K(T_o)$ = \$113.380 per day and, from (16), the optimal order-level of the system is S_o = 200 units.

If deterioration were disregarded, then, as Dave[3] has found, the optimal value of the scheduling period would have been 13 days. In this case, from (11), the total cost of the system would have been \$207.046 per day. Thus, due to the developed model, there will be a reduction of \$93.666 per day in the total cost of the system.

4.THE MODEL WITH LEAD TIME

We, now, extend the inventory model developed in the previous section by including a non-zero lead time, say, L. The order-level, in this section, will be denoted by Z. Consequently, assumption (2) of the previous section, is modified to read

(2) Replenishment rate is infinite. There is a lead time of L time units, which is known and constant during the period under consideration. At the beginning of every scheduling period, which is of T time units long, a variable quantity is ordered so as to achieve the order-level of Z units.

Other assumptions remain unaltered.

First of all, consider the lead time L. If Q_0 denotes the on hand inventory at the time of placing an order, then Q_0 + the quantity on order = Z and $(Q_0 - X)$ is the initial inventory level of the lead time period L, where X is the instantaneous demand during L. Then, reasoning as before, since shortages are not allowed, it is easy to see that we must have

$$Q_0 = b(L) \qquad \text{-----(18)}$$

Similarly, the order-level Z must be

$$Z = b(L + T) \qquad \text{-----(19)}$$

Also, as before, the inventory level of the system during the lead time L is given by

$$Q(t) = (Q_0 - X)(1 - \theta)^t \quad , \quad t = 0,1,2,....,L. \qquad \text{-----(20)}$$

Now, due to the demand and the deterioration of the units during the lead time L, the initial inventory level for the scheduling period T will be less than Z. Hence, from (18) - (20), the expected initial inventory level for the scheduling period T is given by

$$Z_1 = Z - E\{Q_0 - Q(L)\}$$

$$= b(L + T) - b(L)A(L) - RL(1 - \theta)^L \qquad \text{-----(21)}$$

Now, consider the scheduling period T and, note that, the inventory position Q(t), (t = 0,1,2,...,T), of the system during T is given by

$$Q(t) = (Z_1 - X)(1 - \theta)^t \quad , \quad t = 0,1,2,...,T. \qquad \text{-----(22)}$$

Then, from (22), the expected number of units that deteriorate and the expected average amount carried in inventory per time unit of the system during T are

$$D(T) = Z_1 - \mu(T) - E\{Q(T)\} = (Z_1 - RT).A(T) \qquad \text{-----(23)}$$

and

$$I_1(T) = \{1/(T + 1)\}\sum_{t=0}^{t=T} E\{Q(t)\}$$

$$= (Z_1 - RT)A(T + 1)/\theta(T + 1) \qquad \text{-----(24)}$$

respectively, where A(T) is the same as that given by (7). Hence, the expected average total cost per time unit of the system during the period under consideration is

$$K(T) = CD(T)/T + C_1I_1(T) + C_3/T$$

$$= (Z_1 - RT)\{CA(T)/T + C_1A(T + 1)/\theta(T + 1)\} + C_3/T \qquad \text{----(25)}$$

First of all, note that, when there is no lead time, L = 0 implies that $Z = Z_1 = b(T)$ from (19) and (21). Then, the cost function (25) reduces to the cost function (8) of the model with no lead time as developed in the previous section.

Secondly, in the absence of deterioration, $\theta = 0$, and, then, the equations (21) and (25) reduce to

$$Z_1 = b(L + T) - RL \qquad \text{(as expected)} \qquad \text{-----(26)}$$

and

$$K(T) = C_1\{b(L + T) - R(L + T)\} + C_3/T \qquad \text{-----(27)}$$

respectively, which are the same as those given by Dave[3] for a corresponding lead time model for non-deteriorating items.

A Particular Case:-

Letting $b(T) = pRT$ as before, from (21) and (25), we get

$$Z_1 = pRT + (p - 1)RL(1 - \theta)^L \qquad \text{-----(28)}$$

and

$$K(T) = (p - 1)R\{T + L(1 - \theta)^L\}\{CA(T)/T + C_1A(T + 1)/\theta(T + 1)\}$$

$$+ C_3/T \qquad \text{-----(29)}$$

respectively. Then, using (29) and (13) in (12), the condition for optimality at $T = T_o$ is

$$M(T_o) \leq C_3/(p - 1)R \leq M(T_o + 1) \qquad \text{-----(30)}$$

where

$$M(T) = C\theta(T - 1)T(1 - \theta)^{(T-1)}$$

$$+ C_1(T - 1)\{1 - (1 - \theta T^2)(1 - \theta)^T\}/\theta(T + 1)$$

$$+ L(1 - \theta)^L\{CN(T - 1) + C_1(T - 1)N(T)/\theta(T + 1)\} \qquad \text{----(31)}$$

and

$$N(T) = (1 + \theta T)(1 - \theta)^T - 1 \qquad \text{-----(32)}$$

The minimum total cost of the system is obtained by substituting $T = T_o$ in (29) and the optimal order-level of the system is

$$Z_o = pR(L + T_o) \qquad \text{-----(33)}$$

It is easily verifiable that when $L = 0$, this particular case reduces to the particular case of the inventory model without lead time developed before.

Similarly, when $\theta = 0$, from (32), $N(T) = 0$ and, from (31), $M(T) = C_1(T - 1)T$. Then, the optimality condition (30) reduces to (17) and the above model reduces to the similar model for non-deteriorating items given in [3].

At this stage it is interesting to note that unlike the equivalent system for non-deteriorating items as described by Dave[3], where the optimality condition does not depend on the lead time L, here the optimality condition is dependent on the value of L, i.e. here the lead time does play a significant role in determining the optimal scheduling period T_o.

5.EXAMPLE 2

Reconsider example 1 and let the lead time be given by $L = 7$ days. As before, $C_3/(p - 1)R = 8$. For different values of T, the values of the functions N(T), M(T) and K(T) are given in table 2.

Table 2. Values of N(T), M(T) and K(T)

T	A(T)	N(T)	M(T)	K(T)
1	0.015	-0.0002	0.0000	317.668
2	0.030	-0.0007	1.2152	224.459
3	0.044	-0.0013	3.5644	213.312
4	0.059	-0.0022	7.0406	213.160
5	0.073	-0.0032	11.5425	217.182

Applying condition (30), the optimal value of the scheduling period is found to be T_o = 4 days. The minimum total cost of the system is $K(T_o)$ = $213.16 per day and the optimal order-level of the system is Z_o = 550 units.

If deterioration were disregarded, then, as Dave[3] has calculated, the optimal scheduling period of the system would have been 13 days; in which case, the total cost of the system would have been $298.573 per day. Thus, the developed model saves $85.413 per day in the total cost of the system.

6.A SPECIAL CASE OF THE MODEL WITH LEAD TIME

In developing the inventory model with lead time, the lead time of the system is assumed to be of any fixed length L which is a known constant independent of the scheduling period T. On the other hand, the lead time can be any multiple of the scheduling period. A simple example of such a situation is the inventory system in which the ordered lot arrives exactly at the time of placing the next order, i.e. the inventory system with L = T.

Many retail and wholesale businesses operate in this manner. To include all such possibilities, we assume

$$L = nT \qquad \text{-----(34)}$$

where $n \geq 0$ is a known integer. In this case, from (29), the expected average total cost of the system, when $b(T) = pRT$, is given by

$$K(T) = (p - 1)R\{1 + n(1 - \theta)^{nT}\}\{CA(T) + C_1TA(T + 1)/\theta(T + 1)\} + C_3/T \qquad \text{-----(35)}$$

Consequently, the condition for optimality at $T = T_o$ is

$$M(T_o) \leq C_3/(p - 1)R \leq M(T_o + 1) \qquad \text{-----(36)}$$

where

$$M(T) = C\theta(T - 1)T(1 - \theta)^{(T-1)} + C_1(T - 1)\{1 - (1 - \theta T^2)(1 - \theta)^T\}/\theta(T + 1) + n(T - 1)(1 - \theta)^{n(T-1)}\{CTW(T - 1) + C_1(T - 1)W(T)/\theta + C_1(1 - \theta)^n A(T + 1)/\theta(T + 1)\} \qquad \text{-----(37)}$$

and

$$W(T) = (1 - \theta)^n A(T + 1) - A(T) \qquad \text{-----(38)}$$

The optimal order-level, in this case, is given by

$$Z_o = pR(n + 1)T_o \qquad \text{-----(39)}$$

When n = 0, L = 0 and, then, the developed model reduces to the model without lead time.

Also, when $\theta = 0$, from (38), W(T) = 0 and hence, from (37), M(T) = $C_1(n + 1)T(T - 1)$. Then, the optimality condition (36) reduces to

$$(T_o - 1)T_o \leq C_3/C_1(p - 1)R(n + 1) \leq T_o(T_o + 1) \qquad \text{-----(40)}$$

Note that (40) is the same as that derived by Dave[3] for the corresponding model for non-deteriorating items.

7.CONCLUDING REMARKS

For the particular case of b(T) = pRT, when p = 1 = r, the demand is deterministic and the system reduces to the EOQ system in which, due to the demand being instantaneous at the beginning of the scheduling period, inventories are not carried at all and there is only the replenishment cost to be incurred.

Besides b(T) = pRT, another particular case of interest is b(T) =

$(1 + q/T^{1/2})\mu(T) = R(T + qT^{1/2})$, where $q \geq 0$ is a known constant; which in turn imply that $a(T) = (1 + s/T^{1/2})\,\mu(T) = R(T + sT^{1/2})$, where $s \leqslant 0$ is also a known constant. Analysis of this case is similar. Also, when $q = 0 = s$, the demand becomes deterministic and the comment of the preceding paragraph applies.

Further investigation into the system considered in this paper may include a variable rate of deterioration and/or a probabilistic lead time and/or shortages. Another direction would be to develop multi-period models. We leave these possibilities to future efforts.

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