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THE WEBER PROBLEM WITH SUPPLY SURPLUS

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ABSTRACT

In the classical Weber problem the quantities to be transported from the supply and demand point are known. In this paper a problem is considered in which the total available supply exceeds the total demand. In such a situation the optimal selection of points of supply, together with supplied quantities, depends upon the site of the facility. Several heuristics and a branch and bound algorithm for solving this problem are proposed and computational experience is presented.

The problem of finding the optimal location of a central facility in order to serve a set of customers with known demand at minimum cost was first formulated by Weber [1909]. He proposed the following mathematical model for solving this problem: given a set $\{P_1, P_2, \ldots, P_m\}$ of points in a plane representing customers, and for each j the demand d_j of the customer with index j, determine the location of a central facility P from which the demands must be satisfied and for which the total transportation costs are as small as possible. It is assumed that the transportation costs are proportional to the distances covered and to the quantities delivered. The objective is given by

$$\underset{P}{\text{minimize}} \quad \Sigma_{j=1}^{m} d_j \, D(P, P_j) \qquad (1)$$

where $D(P,P_j)$ is the distance between the central facility located at P and customer j. If Euclidean distances are used the problem can be written as:

$$\underset{x,y}{\text{minimize}} \quad \Sigma_{j=1}^{m} d_j \, ((x-x_j)^2 + (y-y_j)^2)^{1/2} \qquad (2)$$

where (x_j, y_j) is the location of customer j and (x,y) the location of the central facility. There is no general analytical formula for finding the optimal point (also called the Weber point). However, in some cases the problem can be trivially solved. For instance, if one of the demands d_j is larger than or equal to the sum of all other demands, then P_j is the Weber point. The inverse result does not hold but weaker conditions have been given for one of the demand points to be the Weber point (see for example Juel and Love [1981].

Also several iterative gradient improvement techniques have been proposed for solving the problem (see the papers by Kuhn and Kuenne [1962], Ostresh [1978], Overton [1983] and Weiszfeld [1937]).

A slight modification of the Weber location problem (W.L.P.) also makes it possible to take supply points into account. If some of the points are the supply points of the goods and if the available quantity is known for each of these points the problem is solved in the same way as the W.L.P. providing that the available quantities are entirely shipped to the central facility.

Quite often however the goods may be obtained from many different suppliers. The total available supply then exceeds the total demand. This demand has only to be met and not exceeded by the supplies, and thus a choice has to be made among the possible suppliers, determining the quantities that will be transported from each supply point. It is reasonable to assume that the most advantageous choice will be made, which in the Weber locational philosophy, will correspond to the nearest supplies. Of course the sites and quantities of supplied goods to be transported will now depend on the location of the central facility, whereas the given data (sites and transported quantities) concerning demands are considered to be fixed.

Let P_j denote the location of customer j ($j=1,\ldots,m$) and d_j the corresponding demand. Let S_i denote the location of candidate supplier i ($i=1,\ldots,k$) and q_i the quantity available at this site.
Let P be the unknown location of the planned central facility. Introducing the variables z_i ($i=1,\ldots,k$), which represent the quantities shipped from supplier i to the central facility, we obtain the following Weber problem with supply surplus (WPSS):

$$\underset{P,z_i(i=1,\ldots,k)}{\text{minimize}} \quad \Sigma_{i=1}^{k} z_i D(S_i,P) + \Sigma_{j=1}^{m} d_j D(P_j,P) \qquad (3)$$

subject to

$$\sum_{i=1}^{k} z_i = d \quad (= \Sigma_{j=1}^{m} d_j) \qquad (4)$$

$$0 \leq z_i \leq q_i \quad i=1,\ldots,k \qquad (5)$$

In this paper we study the WPSS with Euclidean distances. Thus if P_j has coordinates (x_j, y_j), S_i coordinates (x_i', y_i') and P has the unknown coordinates (x,y), the objective (3) becomes

$$\underset{x,y,z_i}{\text{minimize}} \; \Sigma_{i=1}^{k} z_i ((x_i'-x)^2 + (y_i'-y)^2)^{1/2} + \Sigma_{j=1}^{m} d_j ((x_j-x)^2 + (y_j-y)^2)^{1/2} \qquad (6)$$

The paper is organized as follows. In Section 1 several properties of the WPSS model are given, showing that the problem can be solved by considering only a finite number of classical Weber problems. The second section contains several heuristic methods of solution. In Section 3 we develop an exact solution procedure, using a branch and bound approach in order to further reduce the number of Weber problems to be solved. In the final section the efficiencies of the heuristics proposed in Section 2 are compared and some computational results are given for the branch and bound method.

1. Properties

In some cases the WPSS is trivial to solve:

1) If $d = \sum_{i=1}^{k} q_i$ then all available supplies must be shipped to the central facility and the problem becomes a W.L.P.
2) If for each supplier i, $q_i \geq d$ the problem is solved by considering each point i in turn as providing the total supply, disconsidering all other supply points. As in this case the weight of the considered supply point is at least equal to the sum of the weights of the other points (the demand points) the Weber point is located at this supply point. The supply point for which the value of this solution is minimal is then selected as the optimal location.

In the sequel we will suppose that such trivial situations do not occur, or

$$\min_{i=1,\ldots,k} q_i < d < \Sigma_{i=1}^{k} q_i \qquad (7)$$

Considering all solutions of the system:

$$\sum_{i=1}^{k} z_i = d \qquad (8)$$

$$0 \leq z_i \leq q_i \qquad i=1,2,\ldots,k \qquad (9)$$

it can be seen that most of these cannot correspond to an optimal solution of the WPSS. If an optimal solution of the WPSS is denoted by $(P^*;z_i^*,i=1,2,..k)$ then the z_i^* must be the optimal supplies corresponding to a central facility located at P^*. In this solution the supply points nearest to P^* are selected and their entire supply is used until the total demand can be met. This implies the following relationship:

if $z_i^* \neq 0$ and there exists a j $(j \neq i)$ for which $D(P^*,S_j) < D(P^*,S_i)$ then $z_j^* = q_j$, which means that the optimal solution $(z_i^*,i=1,2,..,k)$ defines an extreme point of the convex polytope defined by (8) and (9).

From this relationship the following properties of the optimal solution are easily derived:

Property 1: all but perhaps one of the chosen suppliers will deliver their total offered supply

Property 2: any supply point which is not an extreme point of the convex hull of the used supply points must be fully used (quantity shipped = available supply)

Property 3: if one of the used supply points is not fully used, it must be one of the extreme points of this set.

Property 1 reduces the infinity of possible optimal selections to a finite number. The geometric properties 2 and 3 reduce this number even further as can be seen in the following example.

There are 5 candidate supply points shown in Figure 1. The total demand is 32 and the available quantities at the candidate supply points are all equal to 10.

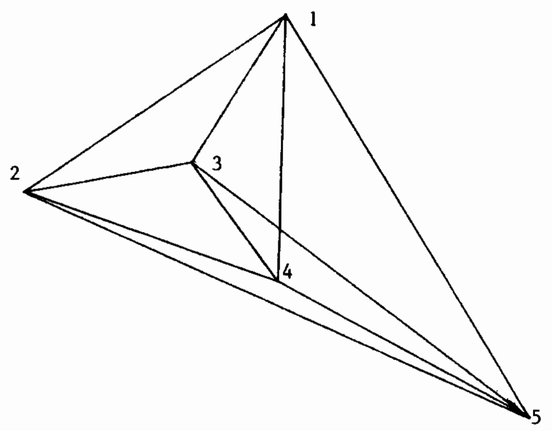

Figure 1. Example used to illustrate the properties of the optimal solution of the WPSS.

Only using property 1, 20 combinations must be considered ($4 \cdot C_5^4$ as 4 suppliers must be used and each of them can be the fractional supplier). With properties 2 and 3 there are:

eliminating point 1: 3 possibilities

eliminating point 2: 4 possibilities

eliminating point 3: 0 possibilities } points cannot be eliminated

eliminating point 4: 0 possibilities } since they lie in the convex hull under consideration

eliminating point 5: 3 possibilities

10 in total

The gain observed depends upon the spatial configuration of the supply points. It can be expected that for increasing numbers of suppliers this gain will become more important. Some indications in this direction are to be found in the work of Lee [1982]. If all available quantities q_i are equal, the problem reduces to finding the $n = [d/q_i] + 1$ nearest supply points ($[x]$ is the integer part of x). Lee showns that the number of possible subsets of supply points which are of the type n-nearest neighbours of a point in the plane is of the order $O(n(k-n))$, which is linear in k for fixed n. Using property 1, this number has still to be multiplied by n, but remains linear in k for fixed n. Taking only property 1 into account, without geometrical considerations, the number of subsets to be considered would be $n\,C_k^n$ which is $O(k^n)$ for fixed $n < \frac{k}{2}$.

2. Heuristics

In this section three heuristic algorithms are proposed for solving the WPSS.

2.1. The greedy approach

For each supply point S_i the sum of distances to all demand points, weighted by the demands, is considered as a measure of the closeness to the set of customers:

$$a_i = \sum_{j=1}^{m} d_j D(S_i, P_j) \qquad \text{for } i=1,\dots,k. \tag{10}$$

The values a_i are ordered by increasing values and suppliers are selected until the total demand is met, the last selected one being the eventual fractional supplier. The resulting WLP is then solved using for example the method of Overton [1983].

2.2. The greedy interactive approach

This method is a variant of the previous heuristic in which the selection of a next supply point is based on the solution of a number of WLP's. In each of these WLP all previous selected supply points are assigned their total available quantity and a single not yet selected supplier supplies the rest. The supplier for which this WLP yields the best solution is added to the selection list.

The steps are:

1. Define $I = \{S_1, S_2, \ldots, S_m\}$ and $I_1 = \emptyset$ (set of selected suppliers)
2. For each $S_i \in I \setminus I_1$ solve the WLP with

$$\begin{cases} (P_j, d_j) \text{ for all customers} \\ (S_t, q_t),\ t \in I_1 \\ (S_i,\ d - \sum_{t \in I_1} q_t) \end{cases}$$

   call the optimal value f_i.
3. Call $f_{i_0} = \min_{i \in I \setminus I_1} f_i$

   If $\sum_{t \in I_1} q_t + q_{i_0} < d$ add i_0 to I_1 and go to step 2

   If not, attribute a supply of $(d - \sum_{t \in I_1} q_k)$ to supplier i_0 and stop.

2.3. The simplified greedy interactive approach

The foregoing heuristic has the great disadvantage that many WLP's have to be solved, which takes quite some time as will be seen in the computational results. To circumvent this difficulty a simplified version was devised in which the selection of a next supply point is based on its proximity to the Weber point found in the previous stage. It consists of the following steps:

1. Define $I = \{S_1,\ldots,S_m\}$ and set $I_1 = \emptyset$

2. Let a_t be the lowest value found in the greedy approach (see section 2.1)

3. Solve the WLP with

   (P_j,d_j) for all customers

   (S_i,q_i) for $i \in I_1$

   $(S_t,\ d - \sum_{i\in I_1} q_i)$

   Call the optimal location W

4. If $q_t + \sum_{i\in I_1} q_i \geq d$ then attribute a supply of $(d - \sum_{i\in I_1} q_i)$ to S_t and stop.

   If $q_t + \sum_{i\in I_1} q_i < d$ add S_t to I_1, rename S_t the supplier of $I\backslash I_1$ closest to W and go to step 3.

2.4. The improvement approach

If P is fixed there remains a trivial selection problem. If the z_i are fixed, the problem is a WLP. The improvement heuristic applies these two steps alternatively after obtaining a first solution with one of the previous methods. It will stop as soon as the selection problem yields the preceding selection. A simplified version consists of not completely solving the WLP, by just carrying out one iteration and then selecting (or attempting to select) new suppliers.

3. A Branch and Bound approach

It can easily be seen that the objective of the WPSS is not a convex function. Even the reduced function in which only the location of the central facility is variable, the choice variables z_i being selected optimally for each location, is not convex. Therefore, a method such as steepest descent will not work and only give a local optimal solution.

In fact, by considering the sets of locations for which the optimal selection of suppliers stay constant, the plane is divided into polygonal regions in which the function becomes convex. An enumeration of all these regions with their associated optimal value of the objective function, will of course yield the optimal solution. However, the difficulty of constructing these regions and their number make this approach unfeasible, except perhaps in the special case discussed at the end of Section 1, applying the method of Lee [1982]. Therefore a branch and bound type approach is proposed in which the selections satisfying properties 1,2 and 3 of Section 1 are enumerated implicitely.

This method requires the calculation of a lower bound. This may be done using the following notations:

J is the set of demand points.

I_0 is the set of eliminated supply points (for which z_i is already set to zero).

I_1 is the set of selected supply points (z_i is set to be non zero).

$I_2 = I \setminus (I_0 \cup I_1)$ is the set of remaining or free supply points.

The following bound can be considered:

If we consider the relaxation obtained by removing the constraints on the available supplies (5), the optimal solution is given by (cfr.Property 2)

$$L = \min_{i \in I_1 \cup I_2} \sum_{j \in J} d_j D(P_j, S_i).$$

Being the optimal value of a relaxed problem, L is a lower bound on all solutions obtained when I is restricted to $I_1 \cup I_2 = I \setminus I_0$. This bound thus only depends on I_0.

The Branch and Bound scheme consists of the following steps:

step 1: Set $I_2 = I$, $I_1 = I_0 = \emptyset$.

step 2: Find a supplier S_t in I_2 with largest total available supply q_t.
Transfer S_t from I_2 to I_1, and compute the new convex hull of I_1 denoted by co I_1.
If $(I_0 \cap \text{co } I_1) \neq \emptyset$ then go to step 5 (i.e. backtrack since property 2 is violated).

step 3: Transfer all points of $(I_2 \cap \text{co } I_1)$ to I_1 (i.e. apply property 2).
Set $q = \sum_{i \in I_2} q_i$ and $s = d - \sum_{i \in I_1} q_i$.
If $s > 0$ and $q \geq s$ go to step 2 (more supply is needed)
If $s \leq 0$ go to step 4 (supply is sufficient)
In all other cases go to step 5 (backtrack)

step 4: If for all extreme points S_e of I_1 $q_e + s < 0$ go to step 5.
For each extreme point S_e of I_1 such that $q_e + s \geq 0$ solve the WLP defined by

$$
\begin{array}{ll}
(P_j, d_j) & j \in J \\
(S_i, q_i) & i \in I_1 \backslash \{e\} \\
(S_e, q_e + s) &
\end{array}
$$

If the value for the objective function is less than the incumbent, update the latter.
If $s \neq 0$ then go to step 2 (perhaps more supply points may lead to a better solution).

step 5: Backtracking.
If $I_1 = \emptyset$ then stop.
Transfer all suppliers from I_1, which were transferred to I_1 during the last instance of step 3, to I_2.
Transfer the last supplier, transferred to I_1 during the last instance of step 2 to I_0.

Adjust co I_1, s and q.

Calculate the lower bound L.

If $q \geq s$ and L is less than the best solution found so far go to step 2.

Otherwise repeat step 5.

Note that other selection strategies could be considered in step 2 of the Branch and Bound algorithm. For example, minimizing the a_i (see eq. 10), adapted to I_2, was suggested by a referee.

4. Computational results

The heuristics and the branch and bound algorithm were implemented in PASCAL and run on a CDC 6600. All data were randomly generated from uniform distributions, the characteristics of which are described below.

27 test problems were run. As the following results will show, further research is needed to increase the efficiency of our lower bound. Indeed, the lower bound presented here was practically ineffective and in 24 out of the 27 test problems only the geometric properties of the optimal solution used in the branch and bound scheme prevented the algorithm from making a complete enumeration of all possible selections, even though the starting solution found with the heuristics was optimal in all cases.

The first group of test problems consisted of 14 problems with 5 demand points, randomly chosen in the square $[10,40]^2$, with random demands in [160,200], and 15 supply points randomly situated in $[0,40]^2$, with random available supplies in [100,200]. The second group consisted of 13 problems with 10 demand points with random demands in [80,100] and 15 supply points with random available supplies in [100,200]; the locations were randomly drawn as before.

The following tables summarize the results obtained using the following notations:

Heuristic 1 denotes the greedy heuristic

Heuristic 2 denotes the greedy interactive heuristic

Heuristic 3 denotes the simplified greedy interactive heuristic

A denotes the results when applying the heuristic alone

I denotes the results when combined with the improvement approach.

% gives the deviation between the optimal value of the objective and the value found when applying the heuristic, expressed in percents of the optimal value

T is the time in milliseconds (in the case of I only the time for the improvement part is given)

Heuristic 4 consists of first determining the Weber point of the demand points, and then applying the improvement approach starting with this Weber point.

BB denotes the branch and bound algorithm starting with the best solution found by the heuristics.

NS is the number of total solutions constructed during the algorithm

NLB is the number of times the lower bound was effective for pruning the BB tree.

With regard to the heuristics, it is clear that the greedy interactive approach has to be rejected in view of its excessive running time. In most cases the simplified version performed just as well, in much less time. It is also clear from these tables that the most powerful tool is the improvement heuristic. The most interesting heuristic seems to be the last one, almost purely improvement.

Concerning the Branch and Bound scheme the most important remark has already been made: we need a much better lower bound. The excessive running time of the algorithm is not only a direct consequence of this lower bound problem, but also of the time needed to solve the WLP, which has to be done repeatedly in step 4.

A suggestion for another lower bound is as follows.

All suppliers of I_l which are not extreme points of the convex hull of I_l have to be chosen at their total available supply. Let us call the set of these points I_l^0, then we will have $\sum_{i \in I_l^0} q_i < d$, since otherwise we would backtrack. Thus the optimal value given by the WLP with

$$(P_j, d_j) \qquad j \in J$$
$$(S_i, q_i) \qquad i \in I_l^0$$

will be a lower bound on all solutions obtained by selection of at least those suppliers belonging to I_l. This bound only depends on I_l.

If in addition $\sum_{i \in I_l} q_i > d$, then all extreme points of the convex hull of I_l with available supply less than or equal to $\sum_{i \in I_l} q_i - d$, will have to be chosen at their total available supply. Indeed, in the optimal solution at most one extreme point can be chosen partially (property 1) and thus it cannot be one of the points described above. In this case, these points can be added to I_l^0, yielding a better lower bound.

A difficulty with this lower bound is that it requires the complete solution of a WLP. The usual algorithms for solving the WLP generate only upper bounds to the optimal value and can therefore not be used. An alternative would be to use an algorithm providing a lower bound to the optimal value (see for example Love and Yeong [1981], Elzinga and Hearn [1983], Wendell and Peterson [1984]).

Problem Nr	Heuristic 1				Heuristic 2				Heuristic 3				Heuristic 4		Branch and Bound		
	A		I		A		I		A		I						
	%	T	%	T	%	T	%	T	%	T	%	T	%	T	T	NS	NLB
1	0	73	0	9	0	4404	0	7	0	468	0	8	0	66	12979	215	0
2	1.96	101	0	87	5.59	3836	5.59	8	5.59	265	5.59	8	0	169	33041	161	0
3	0.22	182	0	108	0	7516	0	8	0	740	0	8	0	108	10293	27	0
4	4.77	77	0	459	0	8019	0	9	0.09	1076	0	286	0	463	25853	197	0
5	1.65	64	0	55	0	5620	0	7	0	393	0	7	0	54	8003	36	0
6	0.91	167	0.77	163	0	7765	0	8	0	531	0	9	0.77	312	7004	90	1
7	0	190	0	7	0	6836	0	7	0	896	0	7	0	178	6594	31	0
8	0	67	0	8	0	4543	0	8	0	518	0	8	0	59	25636	270	0
9	1.14	96	0.92	85	0	4673	0	8	2.71	1100	2.71	7	0.92	84	10804	173	0
10	4.10	62	0	180	0	4506	0	8	0	343	0	9	2.81	88	23232	256	0
11	0	104	0	8	0	4297	0	8	0	964	0	8	0	169	86069	419	6
12	0	148	0	8	0	6557	0	8	0	928	0	7	0	138	12465	165	1
13	1.64	59	0	87	0	4518	0	8	0	532	0	8	0	148	10777	83	0
14	0.33	126	0.33	7	0.33	7127	0.33	8	0	786	0	8	0	118	31815	262	0

Table 1: Computational results for the first set of test problems.

Problem Nr	Heuristic 1				Heuristic 2				Heuristic 3				Heuristic 4		Branch and Bound		
	A		I		A		I		A		I						
	%	T	%	T	%	T	%	T	%	T	%	T	%	T	T	NS	NLB
1	0	122	0	8	0	7284	0	7	0	590	0	7	0	179	47488	250	0
2	12.35	124	0	66	0	4785	0	6	0	362	0	8	0	157	12487	97	00
3	0.14	123	0	94	0	6302	0	8	0	664	0	9	0	166	8561	64	0
4	0	108	0	7	0	8906	0	7	0	631	0	7	0	79	4257	36	0
5	2.42	102	0	119	0	8557	0	8	0	454	0	8	0	185	15266	164	0
6	0.78	146	0	134	0	9736	0	8	0	1198	0	8	0	134	886	10	0
7	0.72	114	0	65	0	16398	0	7	0	1018	0	7	0	63	18837	136	0
8	5.95	123	1.08	278	0	5281	0	8	0	505	0	8	7.42	83	19254	251	0
9	0	691	0	7	0	11338	0	8	0	1474	0	7	0	799	23190	214	0
10	0.23	124	0	111	0	6669	0	7	0	439	0	7	0	202	746	8	0
11	2.18	95	0	105	0	5136	0	8	0	1651	0	8	0	105	49460	424	0
12	7.50	105	0	240	1.85	5411	0	67	1.85	574	0	67	0	297	57124	847	0
13	0.79	364	0	175	0	10696	0	8	0	953	0	8	0	174	23821	186	0

Table 2: Computational results for the second set of test problems.

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