VARIATION IN PARAMETERS OF A QUADRATIC FRACTIONAL FUNCTIONALS PROGRAMMING

S.P. AGGARWAL *

The University of Calgary (Canada).

Introduction.

In practical situations, such as the chicken feed problem, the cost of any individual feed will vary from week to week, similar situations may arise for other elements of the problem. In this paper the resulting affects on an optimal solution of arbitrary variation of any coefficient of a quadratic fractional functional programming are analysed. If the optimal solution of the original problem is known [1], we study the possibility of any method which helps us to solve the new problem which is obtained by varying the parameters in the original problem. In actual existing problems the parameters a_{ij}, b_i, c_j and d_j are either estimates or they vary over the time. Work in this and related area has been done by Shetty [10], Courtillot [4], Garvin [5], Madansky [8], Wagner [12], Saaty [9], Aggarwal [2] and Swarup [11].

Mathematical Model:

$$\text{Maximize } Z = \frac{\left(\sum_{j=1}^{n} c_j x_j + \alpha\right)^2}{\left(\sum_{j=1}^{n} d_j x_j + \beta\right)^2} = \frac{(c'\chi + \alpha)^2}{(d'\chi + \beta)^2} \tag{0.1}$$

subject to

$$\sum_{j=1}^{n} a_{ij} x_j = b_i \quad \text{or} \quad A\chi = b \quad i = 1, 2, ..., m \tag{0.2}$$

$$x_j \geqslant 0 \quad \text{or} \quad \chi \geqslant 0 \quad j = 1, 2, ..., n \tag{0.3}$$

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If we write

$$P_0 = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix} \qquad \text{and} \quad P_j = \begin{bmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{mj} \end{bmatrix}$$

then (0.2) may be written as

$$x_1 P_1 + x_2 P_2 + \ldots + x_n P_n = P_0 \tag{0.4}$$

It is assumed that

(i) The set S of feasible solutions is regular.
(ii) $d'\chi + \beta$ is positive for all feasible solutions.
(iii) Every basic feasible solution is non-degenerate.
(iv) S contains at least two different points.
(v) α and β are given constants.
(vi) a_{ij}, b_i, c_j and d_j are parameters.

Also

(a) A is $m \times n$ matrix, $n > m$.
(b) χ, c, d are $n \times 1$ vectors.
(c) b is $m \times 1$ vector.
(d) prime denotes the transpose of a vector.

Let $\chi'^0 = (x_1^0, x_2^0, \ldots x_m, 0, \ldots 0)$ be the optimum solution of the original probleb (0.1) through (0.3) when a_{ij}, b_i, c_j and d_j are constants and let the optimal basis corresponding to the above optimum solution be P_1, P_2, ..., P_m.

$\therefore$ We have

$$\sum_{i=1}^{m} P_i \, x^0_i = P_0 \tag{0.5}$$

$$x^0_i > 0 \qquad (i = 1, 2, \ldots, m)$$

As the vectors P_1, P_2, ..., P_m are linearly independent

$$\therefore P_j = \sum_{i=1}^{m} u_{ij} \, P_i \qquad (j = 1, 2, \ldots, n) \tag{0.6}$$

We define the quantities [1]

$$z_j^{(1)} = \sum_{i=1}^{m} u_{ij}\, c_i \qquad (j = 1, 2, \ldots, n)$$

$$z_j^{(2)} = \sum_{i=1}^{m} u_{ij}\, d_i \qquad (j = 1, 2, \ldots, n)$$

$$T_1 = \sum_{j=1}^{m} c_j\, x^0_j + \alpha$$

$$T_2 = \sum_{j=1}^{m} d_j\, x^0_j + \beta$$

$$\xi_j = \frac{x^0_r}{u_{rj}}$$

and x'_0 is an optimal solution [1] of the given problem (0.1) through (0.3) if

$$t_j = [T_2 (c_j - z_j^{(1)}) - T_1 (d_j - z_j^{(2)})]$$

$$\times [\xi_j \{T_2 (c_j - z_j^{(1)} + T_1 (d_j - z_j^{(2)})\} + 2\, T_1\, T_2] \leqslant 0 \qquad \text{for all } j$$

or

$$[T_2 (c_j - z_j^{(1)}) - T_1 (d_j - z_j^{(2)})]$$

$$\times [\xi_j \{T_2 (c_j - z_j^{(1)} + T_1 (d_j - z_j^{(2)})\} + 2\, T_1\, T_2] \leqslant 0 \qquad \text{for all } j$$

Change in the Requirement Vector P_0.

We shall now consider the new problem.

$$\text{Maximize } Z = \frac{(\sum_{j=1}^{n} c_j\, x_j + \alpha)^2}{(\sum_{j=1}^{n} d_j\, x_j + \beta)^2} \tag{1.1}$$

subject to

$$P_1\, x_1 + P_2\, x_2 + \ldots + P_n\, x_n = P_0 + \varepsilon \tag{1.2}$$

$$x_j \geqslant 0 \qquad (j = 1, 2, \ldots, n) \tag{1.3}$$

Here the requirement vector P_0 is changed to $P_0 + \varepsilon$, where $\varepsilon' = (\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_m)$. Let B be the square marrix whose column vectors are

$P_1, P_2, \ldots, P_{m-1}$, and P_m, further let B^{-1} be the inverse of the matrix B. Suppose β_i ($i = 1, 2, \ldots, m$) be the ith row vector of B^{-1} i.e.

$$B^{-1} = \begin{bmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_m \end{bmatrix}$$

We have $\varepsilon = BB^{-1}\varepsilon = (P_1, P_2, \ldots, P_m) \begin{bmatrix} \beta_1 \varepsilon \\ \beta_2 \varepsilon \\ \vdots \\ \beta_m \varepsilon \end{bmatrix} = \sum_{i=1}^{m} \beta_i \varepsilon P_i$ (1.4)

Making use of (0.5) in (1.4) we get

$$P_0 + \varepsilon = \sum_{i=1}^{m} P_i x^0_i + \varepsilon = \sum_{i=1}^{m} P_i x^0_i + \sum_{i=1}^{m} \beta_i \varepsilon P_i = \sum_{i=1}^{m} (x^0_i + \beta_i \varepsilon) P_i$$

The change in P_0 affects the values of the basic variables in the optimal solution and we must consider the following two cases

(1) $x^0_i + \beta_i \varepsilon \geqslant 0 \qquad (i = 1, 2, \ldots, m)$
(2) at least one $x^0_i + \beta_i \varepsilon$ is negative.

$\overline{Z}$ the value of the new objective function is given by

$$\overline{Z} = \frac{[\sum_{i=1}^{m} c_i (x^0_i + \beta_i \varepsilon) + \alpha]^2}{[\sum_{i=1}^{m} d_i (x^0_i + \beta_i \varepsilon) + \beta]^2} = \frac{[T_1 + \sum_{i=1}^{m} c_i \beta_i \varepsilon]^2}{[T_2 + \sum_{i=1}^{m} d_i \beta_i \varepsilon[^2} = \frac{\overline{T}_1^2}{\overline{T}_2^2}$$

where

$$\overline{T}_1 = T_1 + \sum_{i=1}^{m} c_i \beta_i \varepsilon$$

$$\overline{T}_2 = T_2 + \sum_{i=1}^{m} d_i \beta_i \varepsilon$$

It may be observed that $z_j^{(1)} - c_j$ and $z_j^{(2)} - d_j$ remain unaffected by the variation in P_0.

Now $\overline{z}_j$ for the new problem is as

$$\overline{z}_j = [\overline{T}_2 (c_j - z_j^{(1)}) - \overline{T}_1 (d_j - z_j^{(2)})]$$
$$\times [\xi_j \{\overline{T}_2 (c_j - z_j^{(1)}) + \overline{T}_1 (d_j - z_j^{(2)})\} + 2\,\overline{T}_1\,\overline{T}_2]$$

Case (1) $P_1, P_2, \ldots, P_m$ is a feasible basis and it will be an optimal basis provided

$$\bar{z}_j \leqslant 0$$

If all $\bar{z}_j$ are not less than or equal to zero, we may start with this new feasible solution and with the help of the method [1] we an get an optimum solution.

Case (2) $(P_1, P_2, \ldots, P_m)$ is not a feasible solution of the new problem. By making certain modifications it is possible to obtain the optimal solution of the new problem. Let $x^0_{i_k} + \beta_{i_k} \varepsilon$ $(i_k = 1, 2, \ldots, l)$ be all negative ones among $x^0_i + \beta_i \varepsilon$ $(i = 1, 2, \ldots, m)$.

We now consider the following modified problem

$$\text{Maximize } \frac{\left[\sum_{j=1}^{n} c_j x_j - M \sum_{i_k=1}^{l} y_{i_k} + \alpha\right]^2}{\left[\sum_{j=1}^{n} d_j x_j + \beta\right]^2}$$

subject to

$$\sum_{j=1}^{n} P_j x_j + \sum_{i_k=1}^{l} (-P_{i_k})\, y_{i_k} = P_0 + \varepsilon$$

$$x_j \geqslant 0 \qquad (j = 1, 2, \ldots, n)$$

$$y_{i_k} \geqslant 0 \qquad (i_k = 1, 2, \ldots, l)$$

y_{i_k} is the set of variables $i_k = 1, 2, \ldots, l$.

M is very large positive quantity.

The optimum solution of the problem (1.1) through (1.3) coincides with the optimum solution of the modified problem. This is established below.

After introducing the new vector $-P_{i_k}$ $(i_k = 1, 2, \ldots, l)$ the following relation is satisfied

$$\sum_{i=1}^{m} P_i (x^0_i + \beta_i \varepsilon) + \sum_{i_k=1}^{l} (-P_{i_k})(-x^0_{i_k} - \beta_{i_k} \varepsilon) = P_0 + \varepsilon$$

$$i \neq i_k \qquad (i = 1, 2, \ldots, l)$$

$$x^0_i + \beta_i \varepsilon \geqslant 0 \qquad i \neq i_k \qquad (i = 1, 2, \ldots, m)$$

$$-x^0_{i_k} - \beta_{i_k} \varepsilon \geqslant 0 \qquad (i_k = 1, 2, \ldots, l)$$

This means that

$$x_i = \delta_i (x^0_i + \beta_i \varepsilon)$$

$$x_j = 0 \qquad (j = m + 1, ..., n)$$

and

$$\delta_i = \begin{cases} 1 & i \neq i_k \ (i = 1, 2, ..., m) \\ -1 & i = i_k \ (i_k = 1, 2, ..., l) \end{cases}$$

is a feasible solution of the modified problem. New we can use the method [1] to find the optimal solution.

Change in the Coefficient Matrix A.

We consider the case where only one column vector P_s is changed to $\bar{P}_s$ and the matrix A is

$$\bar{A} = (P_1, \ P_2, ..., P_{s-1}, \bar{P}_s, ..., P_n)$$

and the new problem is

$$\text{Maximize} \ \frac{(\sum_{j=1}^{n} c_j x_j + \alpha)^2}{(\sum_{j=1}^{n} d_j x_j + \beta)^2} \tag{2.1}$$

subject to

$$\sum_{\substack{j=1 \\ j\neq s}}^{n} P_j x_j + \bar{P}_s x_s = P_0 \tag{2.2}$$

$$x_j \geqslant 0 \qquad j = 1, 2, ..., n \tag{2.3}$$

Again we assume x'^0 to be an optimal solution of the problem (0.1) through (0.3), and corresponding optimal basis is B. Here also we shall discuss the problem in two parts.

(1) $P_s \in B$

(2) $P_s \notin B$.

Case (1). Since $P_1, P_2, ..., P_m$ are linearly independent, we may write [2]

$$\bar{P}_s - P_s = \delta_{1s} P_1 + \delta_{2s} P_2 + \delta_{ss} P_s + ... + \delta_{ms} P_m$$

or

$$\bar{P}_s = \delta_{1s} P_1 + \delta_{2s} P_2 + ... + (1 + \delta_{ss}) P_s + ... + \delta_{ms} P_m \tag{2.4}$$

$$(1 + \delta_{ss} \neq 0)$$

Our aim is to find the conditions which δ_{is} must satisfy so that optimal basis for the new problem remains $P_1, P_2, \ldots, \overline{P}_s, \ldots P_m$. As $P_1, P_2, \ldots, \overline{P}_s, \ldots, P_m$ are linearly independent, therefore using (0.5) in (2.4) we get [3]

$$P_0 = P_1 \left(x^0_1 - \frac{x^0_s \delta_{1s}}{1 + \delta_{ss}}\right) + P_2 \left(x^0_2 - \frac{x^0_s \delta_{2s}}{1 + \delta_{ss}}\right) + \ldots$$

$$+ \overline{P}_s \left(\frac{x^0_s}{1 + \delta_{ss}}\right) + \ldots + P_m \left(x^0_m - \frac{x^0_s \delta_{ms}}{1 + \delta_{ss}}\right)$$

For the new basis to be feasible, we require

$$x_i - \frac{x^0_s \delta_{is}}{1 + \delta_{ss}} \geqslant 0 \qquad i \neq s \tag{2.5}$$

$$\frac{x^0_s}{1 + \delta_{ss}} \geqslant 0 \tag{2.6}$$

As $x^0_s > 0$, therefore, $1 + \delta_{ss} > 0$ (2.7)

Using (0.6) and (2.4) in this new problem we get

$$P_j = \overline{u}_{ij} P_1 + \overline{u}_{2j} P_2 + \ldots + \overline{u}_{sj} \overline{P}_s + \ldots + \overline{u}_{mj} P_m$$

where $\overline{u}_{ij} = u_{ij} - \dfrac{\delta_{is} u_{sj}}{1 + \delta_{ss}}$

$$\overline{z}_j^{(1)} = z_j^{(1)} - \frac{u_{sj}}{1 + \delta_{ss}} \sum_{i=1}^{m} c_i \delta_{is}$$

$$\overline{z}_j^{(2)} = z_j^{(2)} - \frac{u_{sj}}{1 + \delta_{ss}} \sum_{i=1}^{m} d_i \delta_{is}$$

$$\overline{T}_1 = T_1 - \frac{x^0_s}{1 + \delta_{ss}} \sum_{i=1}^{m} c_i \delta_{is}$$

$$\overline{T}_2 = T_2 - \frac{x^0_s}{1 + \delta_{ss}} \sum_{i=1}^{m} d_i \delta_{is}$$

The new basis is optimal if

$$\overline{l}_j = [\overline{T}_2 (c_j - \overline{z}_j^{(1)}) - \overline{T}_1 (d_j - \overline{z}_j^{(2)})]$$

$$\times [\delta_j \{\overline{T}_2 (c_j - \overline{z}_j^{(1)}) + \overline{T}_1 (d_j - \overline{z}_j^{(2)})\} + 2\, \overline{T}_1 \overline{T}_2] \leqslant 0 \tag{2.8}$$

$$= [(T_2 - \frac{x^0_s}{1 + \delta_{ss}} \sum_{i=1}^{m} d_i \, \delta_{is}) \, (c_j - z_j^{(1)} + \frac{u_{sj}}{1 + \delta_{ss}} \sum_{i=1}^{m} c_i \, \delta_{is})$$

$$- (T_1 - \frac{x^0_s}{1 + \delta_{ss}} \sum_{i=1}^{m} c_i \, \delta_{is}) \, (d_j - z_j^{(2)} + \frac{u_{sj}}{1 + \delta_{ss}} \sum_{i=1}^{m} d_i \, \delta_{is})]$$

$$= [\xi_j \, \{(T_2 - \frac{x^0_s}{1 + \delta_{ss}} \sum_{i=1}^{m} d_i \, \delta_{is}) \, (c_j - z_j^{(1)} + \frac{u_{sj}}{1 + \delta_{ss}} \sum_{i=1}^{m} c_i \, \delta_{is})$$

$$+ (T_1 - \frac{x^0_s}{1 + \delta_{ss}} \sum_{i=1}^{m} c_i \, \delta_{is}) \, (d_j - z_j^{(2)} + \frac{u_{sj}}{1 + \delta_{ss}} \sum_{i=1}^{m} d_i \, \delta_{is})$$

$$+ 2 \, (T_1 - \frac{x^0_s}{1 + \delta_{ss}} \sum_{i=1}^{m} c_i \, \delta_{is}) \, (T_2 - \frac{x^0_s}{1 + \delta_{ss}} \sum_{i=1}^{m} d_i \, \delta_{is})] \leqslant 0$$

In case (2.8) does not hold for all $j = m + 1, m + 2, ..., n$ but basis is feasible we can obtain the optimal solution by using the technique [1] given by author.

If the solution is not feasible we may consider the modified problem whose optimal is the same as that of (2.1) through (2.3).

The modified problem is

$$\text{Maximize} \; \frac{(\sum_{\substack{j=1 \\ j \neq s}}^{n} c_j \, x_j + c_s \, \bar{x}_s + \alpha - M \, x_s)^2}{(\sum_{\substack{j=1 \\ j \neq s}}^{n} d_j \, x_j + d_s \, \bar{x}_s + \beta)^2}$$

subject to

$$P_1 \, x_1 + P_2 \, x_2 + ... + P_s \, x_s + \bar{P}_s \, \bar{x}_s + ... + P_n \, x_n = P_0$$

$$x_j \geqslant 0 \qquad j = 1, 2, ..., n$$

$$\bar{x}_s \geqslant 0$$

M is sufficiently large positive number.

Now $\bar{P}_s = B\bar{B}' \bar{P}_s = \sum_{i=1}^{m} (\beta_i \, \bar{P}_s) \, P_i$.

Obviously B is a feasible basis of the modified problem and the optimal solution can be obtained.

Case (2). Again B is a feasible basis for the new problem but t_s will be affected, say it become $\bar{t}_s$. If all $\bar{t}_s \leqslant 0$ then present basis is optimal. For $t_s > 0$ we still solve the problem till the optimal solution is obtained.

Change in the Coefficient Vector c of the Objective Function.

When we change the coefficient vector c of the objective function, the new objective function becomes

$$\text{Maximize } \bar{Z} = \frac{[\sum_{j=1}^{n} (c_j + \Delta c_j)\, x_j + \alpha]^2}{(\sum_{j=1}^{n} d_j\, x_j + \beta)^2}$$

The original optimal solution remains unaffected but it is not essential that it is optimal for the new problem. Of course $z_j^{(2)} - d_j$ remains unchanged but $z_j^{(1)} - c_j$ changes and

$$\bar{T}_1 = T_1 + \sum_{j=1}^{m} \Delta c_j\, x^0_j\,; \qquad \bar{z}_j^{(1)} = \sum_{i=1}^{m} u_{ij}\,(c_i + \Delta c_i)$$

$$\therefore\ \bar{t}_j = [T_2\,(c_j + \Delta c_j - \bar{z}_j^{(1)}) - \bar{T}_1\,(d_j - z_j^{(2)})]$$

$$[\xi_j\,\{T_2\,(c_j + \Delta c_j - \bar{z}_j^{(1)}) + \bar{T}_1\,(d_j - z_j^{(2)})\} + 2\,\bar{T}_1\,\bar{T}_2]$$

$$= [T_2\,(c_j + \Delta c_j - \sum_{i=1}^{m} u_{ij}\,(c_i + \Delta c_i)) - (T_1 + \sum_{j=1}^{m} \Delta c_j\, x^0_j)\,(d_j - z_j^{(2)})]$$

$$\times [\xi_j\,\{T_2\,(c_j + \Delta c_j - \sum_{i=1}^{m} u_{ij}\,(c_i + \Delta c_i)) + (T_1 + \sum_{j=1}^{m} \Delta c_j\, x^0_j)\,(d_j - z_j^{(2)})\}$$

$$+ 2\,(T_1 + \sum_{j=1}^{m} \Delta c_j\, x^0_j)\ T_2] \tag{2.9}$$

If all $\bar{t}_j \leqslant 0$ $(j = 1, 2, ..., n)$ then the original optimal solution is also optimal for the new problem. If there is some $\bar{t}_j$ is positive, we apply the technique [1] to this new problem to get optimal solution.

If the changes are made in the coefficients of the non-basic variables in the numerator of the objective function, then (2.9) reduces to

$$
\begin{aligned}
\bar{t}_j &= [T_2 (c_j + \Delta c_j - z_j^{(1)}) - (d_j - z_j^{(2)}) T_1] \\
&\quad \times [\xi_j \{T_2 (c_j + \Delta c_j - z_j^{(1)}) + (d_j - z_j^{(2)}) T_1\} + 2 T_1 T_2] \\
&= t_j + T_2^2 \Delta c_j [\xi_j \{\Delta c_j + 2 (c_j - z_j^{(1)})\} + 2 T_1]
\end{aligned}
$$

This shows that if

$$\Delta c_j [\xi_j \{\Delta c_j + 2 (c_j - z_j^{(1)})\} + 2 T_1]$$

$$j = m + 1, m + 2, \ldots, n$$

are negative then original optimal remains optimal to the new problem as well. The change in vector d may be discussed in the same way as for change in vector c.

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