Sampled autocorrelations from an integrated ARMA process

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Abstract

We investigate the behaviour of the lag–k sample serial correlation $r_k^{(n)}$ $(0 < k < n)$, for series realisations (of length n) from any autoregressive integrated moving average (ARIMA) model, and make comparisons with earlier results reported by Hasza (1980).

Keywords : Approximation bias, ARUMA models, Bartlett's formula, centred and non–centred serial correlations, finite–series versus asymptotic approximations, serial covariance, simplifying operators, simulation, typical serial correlation curves

1. INTRODUCTION

Consider a time series realisation of length n, denoted by $\{z_1, \ldots, z_n\}$, and define its k-th serial covariance as

$$c_k^{(n)} = \frac{1}{n} \sum_{i=1}^{n-k} (z_i - \bar{z})(z_{i+k} - \bar{z}) \qquad (k = 0, 1, \ldots, n-1)$$

where $\bar{z} = (z_1 + \ldots + z_n)/n$ is the sample mean of the observed series. Define the k-th serial correlation by

$$r_k^{(n)} = c_k^{(n)} / c_0^{(n)} .$$

In both these relations, the superfix (n) serves to emphasise the length of the series.

Box and Jenkins (1976) have greatly popularised the use of ARMA(p, q) and ARIMA(p, d, q) processes for modelling time series. These are defined by $\{Z_i\}$ satisfying

$$(1 - \phi_1 B - \ldots - \phi_p B^p)(1 - B)^d Z_i = (1 - \theta_1 - \ldots - \theta_q B^q) A_i \qquad (1)$$

with d = 0 for the ARMA cases and $d \geq 1$ for those ARIMA models which are excluded from the ARMA class. In (1), $(\phi_1, \ldots, \phi_p)$ and $(\theta_1, \ldots, \theta_q)$ are two sets of real parameters, with the first subject to the stationarity condition, namely that the polynomial,

$$\phi_p(\zeta) \equiv 1 - \phi_1 \zeta - \ldots - \phi_p \zeta^p$$

in the complex variable ζ, has no zero within or on the unit circle; and B is the backshift operator, such that B^j operating on any X_i, for instance A_i or Z_i, produces X_{i-j}. $\{A_i\}$ is a white noise sequence of independent but identically distributed normal zero-mean random variables, all with variance σ^2 say. Note that it is unnecessary (for our purposes) to impose any further restrictions on the real θ-parameters.

In this paper, we will wish to extend the ARMA class to the so-called ARUMA models, obtained by replacing the d-times differencing operator $(1 - B)^d$, in (1), with

$$U_d(B) = 1 - u_1 B - \dots - u_d B^d,$$

all of whose zeros lie precisely on the unit circle, but not necessarily all (or any) taking the value 1. We then study certain serial dependence properties for the whole ARUMA class. In particular, we consider the first two moments of $c_k^{(n)}$ and $r_k^{(n)}$.

Hasza (1980) has discussed the asymptotic distribution of the sampled serial correlations for ARIMA(p, 1, q) models as $n \to \infty$. His theory is rigorous and provides <u>inter alia</u> formulae for the population mean and variance of a finite-lagged serial correlation from an infinitely long realisation, namely

$$E[r_k^{(\infty)}] = 1 \tag{2}$$

and

$$\mathrm{Var}[r_k^{(\infty)}] = 0, \tag{3}$$

in agreement with Roy and Lefrançois (1978).

However, when application is made to finite-lengthed series realisations, approximations of unknown validity are introduced, and Hasza concentrates on just the mean and variance of the $r_k^{(n)}$ for the ARIMA(0, 1, 1) [or, more briefly, IMA(1, 1)] case,

$$(1 - B)Z_i = (1 - \theta B)A_i \qquad |\theta| < 1; \tag{4}$$

for which he gets

$$E[r_k^{(n)}] \simeq E(n, k, \theta) = 1 - n^{-1}\{7.335k+10.652\theta/(1-\theta)^2\} \tag{5}$$

and

$$\begin{aligned} \mathrm{Var}[r_k^{(n)}] &\simeq S^2(n, k, \theta) \\ &= n^{-2}\{19.501k^2+69.588k\theta/(1-\theta)^2+69.420\theta^2/(1-\theta)^4\}. \end{aligned} \tag{6}$$

In this paper, we advocate an alternative approach to predicting finite series behaviour, which also gives rise to formulae for $E[r_k^{(n)}]$ and $\mathrm{Var}[r_k^{(n)}]$. Again, approximations are employed, but we would suggest that this does not make the theory any less rigorous than Hasza's, for the practical purpose of gaining insight as to how real series behave. Moreover, these formulae appear superior to Hasza's for short series, and are certainly in better agreement with the simulations that he gave to support his theoretical results.

In what follows, γ_k will always denote the k-th theoretical autocovariance for that ARMA part of the model which remains after any I or U factor has been removed, by appropriate simplification, from what was originally an ARMA, ARIMA or ARUMA process.

Finally, we define

$$E^{(n)}_{k,1} = E[c^{(n)}_k] / E[c^{(n)}_0] \tag{7}$$

which can be considered as a first approximation to $E[r^{(n)}_k]$ [with an $O(n^{-1})$ error] - compare Wichern (1973) - and its second approximation, correct to order n^{-1} [with an error of $O(n^{-2})$, and not $O(n^{-3/2})$ as is frequently assumed - see Anderson (1990a)],

$$E^{(n)}_{k,2} = E^{(n)}_{k,1}\left\{1 - \frac{Cov[c^{(n)}_k, c^{(n)}_0]}{E[c^{(n)}_k]E[c^{(n)}_0]} + \frac{Var[c^{(n)}_0]}{E^2[c^{(n)}_0]}\right\}; \tag{8}$$

and, correct to order n^{-1}, an approximation for $Var[r^{(n)}_k]$, viz:

$$V^{(n)}_{k,2} = \{S^{(n)}_{k,2}\}^2$$

$$= \{E^{(n)}_{k,1}\}^2\left\{\frac{Var[c^{(n)}_k]}{E^2[c^{(n)}_k]} - \frac{2Cov[c^{(n)}_k, c^{(n)}_0]}{E[c^{(n)}_k]E[c^{(n)}_0]} + \frac{Var[c^{(n)}_0]}{E^2[c^{(n)}_0]}\right\}.^{*} \tag{9}$$

[In (8) and (9), we can of course replace all the Cov(x, y) and Var(x) by, respectively, E(x)E(y) and $E(x^2)$ - should that be more convenient.]

2. EXACT COVARIANCE RESULTS

Explicit formulae for $E[c^{(n)}_k]$ or $E^{(n)}_{k,1}$ have been obtained for all ARMA and ARIMA models by Anderson (1979a), who also gave the results for the other ARUMA cases in Anderson (1979b). Similar results for $Var[c^{(n)}_k]$ and $Cov[c^{(n)}_k, c^{(n)}_0]$ $(0 \leq k < n)$, given an

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* $V^{(n)}_{k,1} = 0$, rather uninterestingly.

ARMA or ARIMA(p, 1, q), have been recorded in Anderson and De Gooijer (1983) and, for other general homogeneously nonstationary ARUMA models, in Anderson (1980a). Finally, Anderson and De Gooijer (1982, 1988) give $\mathrm{Cov}[c_k^{(n)}, c_h^{(n)}]$ $(0 \le h \le k < n)$, whilst Anderson and De Gooijer (1985) show how similar results can be obtained for space-time systems.

Thus, for ARMA(p, q),

$$E[c_k^{(n)}] = \frac{1}{n^3}\{n(n-k)(n\gamma_k-\gamma_0) + 2n\sum_{j=1}^{k-1}(k-j)\gamma_j - 2n\sum_{j=1}^{n-k-1}(n-k-j)\gamma_j - 2k\sum_{j=1}^{n-1}(n-j)\gamma_j\}. \quad (10)$$

Again, for any ARIMA(p, 1, q) process, we have that

$$E[c_k^{(n)}] = \frac{1}{6n^3}[n(n-k)\{(n^2-4kn+2k^2-1)\gamma_0 - 6n\sum_{j=1}^{k-1}(k-j)\gamma_j - 2n\sum_{j=1}^{k-2}(k-j)\{(k-j)^2-1\}\gamma_j + 2n\sum_{j=1}^{n-k-2}(n-k-j)\{(n-k-j)^2-1\}\gamma_j + 2k\sum_{j=1}^{n-2}(n-j)\{(n-j)^2-1\}\gamma_j] \quad (11)$$

and, in particular, for model (4) this reduces to

$$E[c_k^{(n)}] = (n-k)\{(n\delta_k-1)\theta+(n^2-4kn+2k^2-1)(1-\theta)^2/6)\frac{\sigma^2}{n^2} \quad (12a)$$

where

$$\delta_k = \begin{cases} 1 & (k = 0) \\ 0 & (k \neq 0). \end{cases} \quad (12b)$$

This agrees with formulae given by Roy (1977) - and yields the result, for $E_{k,1}^{(n)}$, previously deduced by Wichern (1973).

Given any ARMA(p, d, q) with d > 1, we find that

$$E_{k,1}^{(n)} = (n-k)(n^2-2nk-2k^2-1)/\{n(n^2-1)\}; \tag{13}$$

whilst, for any ARUMA model of the form (1) with $(1 - B)^d$ replaced by (1 + B), we get

$$E_{k,1}^{(n)} = \{n^2(n-k)(-1)^k-n\delta_{n,k}-k\delta_{n,k}^2\}/(n^3-n\delta_{n,0}) \tag{14a}$$

where

$$\delta_{n,k} = \{n \pmod 2\}(-1)^k = \begin{cases} 0 & (n \text{ even}) \\ 1 & (n \text{ odd, } k \text{ even}) \\ -1 & (n \text{ odd, } k \text{ odd}). \end{cases} \tag{14b}$$

If, instead, we replace the $(1 - B)^d$ by $(1 - 2B\cos\omega + B^2)$; then, with an error of $O(n^{-2})$, we have that

$$E_{k,1}^{(n)} \simeq (1 - \frac{k}{n})\cos k\omega. \tag{15}$$

The corresponding second moment formulae, for ARMA and ARIMA(p, 1, q) models, are much more involved; but they have the simple form

$$\mathrm{Var}[c_k^{(n)}] = \sum_{i=0}^{n-1}\sum_{j=0}^{n-1} g_{i,j}\gamma_i\gamma_j,$$

where the $g_{i,j}$ coefficients are all known. Similarly, for $\mathrm{Cov}[c_k^{(n)}, c_0^{(n)}]$ - see Anderson and De Gooijer (1983) - or, more generally, for $\mathrm{Cov}[c_k^{(n)}, c_h^{(n)}]$, which subsumes both the previous results and was more recently derived in Anderson and De Gooijer (1988). Those for other ARUMA models are less complicated.

For instance, the (1 + B) case yields

$$\frac{\mathrm{Var}[c_k^{(n)}]}{E^2[c_k^{(n)}]} = \frac{2[n^4(n-k)^2 - 2\{n^2(n^2-k^2)-nk\}\delta_{n,k} + (n^2+k^2)\delta^2_{n,k}]}{\{n^2(n-k)(-1)^k - n\delta_{n,k} - k\delta^2_{n,k}\}^2} \tag{16}$$

and

$$\frac{\mathrm{cov}[c_k^{(n)}, c_0^{(n)}]}{E[c_k^{(n)}]E[c_0^{(n)}]} = \frac{2[n^2(n-k)\{n^2(-1)^k - \delta_{n,k}\} + (1-n^2)(n\delta_{n,k} + k\delta^2_{n,k})]}{\{n^2(n-k)(-1)^k - n\delta_{n,k} - k\delta^2_{n,k}\}(n^2 - \delta_{n,0})}. \tag{17}$$

So, on simplifying for the various special cases, (16) and (17) give

$$\mathrm{Var}[c_k^{(n)}] / E^2[c_k^{(n)}] = \begin{cases} 2 + 8n^3/\{(n-k)(n^2-1)^2\} & \text{(n odd, k odd)} \\ 2 & \text{(otherwise)} \end{cases} \tag{18}$$

and

$$\mathrm{cov}[c_k^{(n)}, c_0^{(n)}] / E[c_k^{(n)}]E[c_0^{(n)}] = 2 \quad \text{(all n and k)}. \tag{19}$$

Then, using (7), (8) and (9), we can obtain approximations to $E[r_k^{(n)}]$ and $\mathrm{Var}[r_k^{(n)}]$. Thus, for instance, with the (1 + B) ARUMA we have

$$E_{k,2}^{(n)} = E_{k,1}^{(n)} = \begin{cases} (-1)^k(1-k/n) - 2k/\{n(n^2-1)\} & \text{(n odd, k even)} \\ (-1)^k(1-k/n) & \text{(otherwise)} \end{cases} \tag{20}$$

and

$$V_{k,2}^{(n)} = \begin{cases} 8n(n-k)/(n^2-1)^2 & \text{(n odd, k odd)} \\ 0 & \text{(otherwise).} \end{cases} \tag{21}$$

Since the approximations, $E_{k,2}^{(n)}$ and $V_{k,2}^{(n)}$ are both correct to $O(n^{-1})$, it follows that, for this ARUMA subclass,

$$E[r_k^{(n)}] = (-1)^k (1-k/n) + O(n^{-2}) \tag{22}$$

and

$$Var[r_k^{(n)}] = 0 + O(n^{-2}). \tag{23}$$

Thus, actually observed serial correlations will be given by

$$r_k^{(n)} = (-1)^k (1-k/n) + O_p(n^{-1}). \tag{24}$$

Using $Cov[c_k^{(n)}, c_h^{(n)}]$, we can similarly obtain $Cov[r_k^{(n)}, r_h^{(n)}]$ approximately.

3. AGREEMENT WITH SIMULATION

The theory of the last section gives rise to approximations which provide good agreement with simulation in all the cases studied so far. For instance, see Anderson and De Gooijer (1979, 1980, 1992) and Anderson (1990b).

In particular, using our formulae, we get much better agreement with the Monte Carlo results, based on 1000 replications and reported by Hasza (1980), than he did with his asymptotic approximations. Thus, when $n = 50$ and $\Theta = \pm .8$, we have comparisons for $k = 3$, as shown in Table 1. Note that, there, the exact values quoted are those obtained from numerical integration by a method discussed in Anderson and De Gooijer (1980) and De Gooijer (1980). Also, we have written the results for our $E_{3,1}$ within brackets, as we would not in fact use the first approximation for this model. [On its own, $1 - B$ is an anomalously weak non-stationarity operator, which gives rise to models that behave similarly in some respects to stationary ones. See Anderson (1990), say, for a discussion.]

Table 1

Comparison between Hasza's and our Results for $(1-B)Z_i = (1-\theta B)A_i$

Formulae Compared	$\theta = -.8$	$\theta = .8$
Hasza's observed $\bar{r}_3^{(50)}$	.65	.16
Hasza's $E(50, 3, \theta)$	.61	-3.70^*
[Our $E_{3,1}^{(50)}$	.74	.20]
Our $E_{3,2}^{(50)}$	.65	.15
Exact $E[r_3^{(50)}]$	.66	.17
Hasza's observed s.e.$[r_3^{(50)}]$	.17	.19
Hasza's $S(50, 3, \theta)$	.23	3.58^*
Our $S_{3,2}^{(50)}$	.13	.21
Exact s.d.$[r_3^{(50)}]$	.16	.18

* Impossible values.

Clearly, our formulae do substantially better than Hasza's for $\theta = -.8$, and still perform well in the more testing case of θ = .8, for which Hasza's formulae produce not only very poor but actually impossible values (no serial correlation can have an expectation reaching, let alone exceeding, 1 in magnitude or a standard error as great or greater than 1).

A table comparing results for all of Hasza's length-50 series simulations is given in the Appendix.

4. DISCUSSION

Table 2

Percentages of First Serial Correlations falling within various Ranges for Legth-50 Realisations from $(1-B)Z_i = (1-\Theta B)A_i$

Range of $r_1^{(50)}$	Percentage of Distribution
$\leq$ -.05	6.65
(-.05, .00]	4.54
(.00, .05]	6.27
(.05, .10]	7.98
(.10, .15]	9.40
(.15, .20]	10.31
(.20, .25]	10.55
(.25, .30]	10.12
(.30, .35]	9.12
(.35, .40]	7.68
(.40, .45]	6.09
(.45, .50]	4.50
> .50	6.90

Also note that, although the asymptotic distribution of, say, $r_1^{(n)}$ for the IMA(1, 1) model with $\theta = .8$, is non-normal (as was demonstrated by Hasza, 1980), the short series distributions are not highly negative skew, as intuition might suggest. For instance, when n = 50, the exact probability distribution function is almost symmetric (perhaps even slightly positive skew) as is shown by Table 2 (the expectation being .2151). Also see Anderson and De Gooijer (1980, 1992), where slight (but not significant) positive skewness was noted, for a simulation with 1000 replications.

Again, the typical shape for the run of serial correlations, $\{r_k^{(n)} : k = 1, \ldots , n-1\}$, from a finite series realisation of an ARIMA(p, d, q), with d > 0, is not a slow linear decline - as, for instance, suggested by Box and Jenkins (1976). Rather, it is a smooth catenary-like curve that starts with positive values which decrease to a negative minimum and then increase again towards zero. Compare (12) or (13). That something like this must happen is also indicated by noting that, for any series realisation whatsoever of length n > 1,

$$\sum_{k=1}^{\infty} r_k^{(n)} = -\frac{1}{2} \qquad (25)$$

where $r_k^{(n)}$ is taken as zero whenever $k \geq n$. [A proof is given on p. 736 of Anderson (1990b).]

Finally, the motivation for our study stems from a general belief that, for observed series, the finite sample behaviour of the serial correlations can be very different from what asymptotic theory has previously led practitioners to expect. Thus, we believe that our results provide us with a means of recognising specific ARUMA models, when they occur; and then, by a method analogous to differencing d-times for the ARIMA case, give us a way of reducing the process to just its ARMA part, by application of the appropriately indicated simplifying operator. For instance, see Anderson (1980b).

POSTSCRIPT

A different approach for obtaining approximations to the moments of the serial correlations, for finite series realisations from stationary ARMA(p, q) models, has been discussed by Davies and Newbold (1980) and Anderson and De Gooijer (1988). Davies and Newbold obtained (with some errors) $E[r_k^{(n)}]$ and $Var[r_k^{(n)}]$ approximately, for an ARMA(0, q) [abbreviated to MA(q)] process, when $q \ll n$. Their mean contained a fundamental error, whilst their variance formulae (for various lags k) included six algebraic mistakes. By first extending their method and then using a rather different means, Anderson and De Gooijer (1988) deduced the analogous general formulae, given any ARMA(p, q) model, and also quoted the corresponding $Cov[r_k^{(n)}, r_h^{(n)}]$ result, for the MA(q) case (op. cit., equation 26). However, although the second-moment algebra had been corrected, the fundamental error in the first moment was not rectified until Anderson (1994), who also showed that it luckily cancelled out in the centered second-moment formulae, and further deduced a concise covariance approximation for general ARMA(p, q) and general lag k. In fact, it was the proof-checking of the remaining part of this paper that prompted Anderson (1994), and has lead to some appropriate modification of the material that follows.

Here, we outline the derivation of this general covariance formula (which of course specialises to $\mathrm{Var}[r_k^{(n)}]$).

For simplicity, we drop the superscript (n) and we also work with the non-centred serial correlations for both the observed $\{z_i\}$ and its driving shocks $\{a_i\}$, namely

$$r_k(z) = \sum_{i=1}^{n-k} z_i z_{i+k} \Big/ \sum_{i=1}^{n} z_i^2 \quad (0 < k < n)$$

$$r_k(a) = \sum_{i=1}^{n-k} a_i a_{i+k} \Big/ \sum_{i=1}^{n} a_i^2 \quad (0 < k < n).$$

First we note some results for the $r_k(a)$, which tidy up work reported in Davies (1977). For $0 < k < n$, and m any positive integer:

$$E[r_k^{2m-1}(a)] = 0$$

and

$$E[r_k^{2m}(a)] = O(n^{-m});$$

with, in particular,

$$E[r_k^2(a)] = \frac{(n-k)}{n(n+2)}$$

and

$$E[r_k^4(a)] = \frac{3\{(n-k)^2+2(n-k)+4\langle n-2k\rangle\}}{n(n+2)(n+4)(n+6)}$$

where $\langle x\rangle = x$, if $x > 0$, and is zero otherwise. Finally

$$E[r_j^1(a)r_k^m(a)] = O(n^{-2})$$

or less, whenever $l > 0$. For instance, should l or m (or both of them) be odd, then

$$E[r_j^l(a) r_k^m(a)] = 0 \quad (j \neq k)$$

and

$$E[r_j^2(a) r_k^2(a)] = \frac{(n-j)(n-k)+4(n-k)+8\langle n-j-k\rangle}{n(n+2)(n+4)(n+6)} \quad (j < k).$$

Thus, we see that all powers, other than squares, of the $r_k(a)$'s, and all proper products of powers of the $r_j(a)$'s and $r_k(a)$'s ($j \neq k$), only give rise to terms of $O(n^{-2})$ or less, on taking expectations. [Actually, this is also true for any product of the serial correlations other than the simple single-lag squares, $r_k^2(a)$ $(0 < k < n)$. See Anderson and Chen (1994).]

Then, following Davies and Newbold (1980), we can write

$$r_k(z) \simeq \{1+2\sum_{j=1}^{n-1} \rho_j r_j(a)\}^{-1} \{r_k(a) + \sum_{j=1}^{n-1} \rho_j [r_{k+j}(a) + r_{k-j}(a)]\};$$

where $\rho_j = \gamma_j / \gamma_0$ denotes the lag-j theoretical autocorrelation. So, ignoring powers or cross products with total degree greater than 1 (other than single-lag squares) which will eventually (on taking expectations later) only yield $O(n^{-2})$ or less, we have first that

$$r_k(z) \simeq \{1-2\sum_{j=1}^{n-1} \rho_j r_j(a) + 4\sum_{j=1}^{n-1} \rho_j^2 r_j^2(a)\}\{r_k(a) + \sum_{j=1}^{n-1} \rho_j [r_{k+j}(a) + r_{k-j}(a)]\}$$

and, on assuming* that adding $\sum_{j=0}^{k-1} \rho_{n+j} r_{k-n-j}(a)$ to the terms in the second main bracket, here, will make no difference to final results which have $O(n^{-2})$ errors, we then get

$$r_k(z) \simeq \{1-2\sum_{j=1}^{n-1}\rho_j r_j(a)+4\sum_{j=1}^{n-1}\rho_j^2 r_j^2(a)\}\{\rho_k+\sum_{j=1}^{n-1}(\rho_{k+j}+\rho_{k-j})r_j(a)\}$$

$$\simeq \rho_k + \sum_{j=1}^{n-1}F_{k,j}\, r_j(a) - 2\sum_{j=1}^{n-1}\rho_j F_{k,j}\, r_j^2(a) \qquad (26)$$

where

$$F_{k,j} = \rho_{k+j} + \rho_{k-j} - 2\rho_k\rho_j.$$

[Note that the asterisked assumption is very weak. For instance, it is sufficient that $k\rho_n\rho_{n-k}$ is $O(n^{-1})$ for the assumption to be valid. In fact the assumption can be avoided, if we agree to replace the theoretical correlations, ρ_j, by zero, whenever $|j| \geq n$; which just requires that the process is MA(q) with $q < n$.]

Now, Davies and Newbold (1980) implicitly assumed that the error in (26) was $O(n^{-2})$. However, we can actually show that it is $\eta_k + O(n^{-2})$, where η_k is $O(n^{-1})$ with an expectation of $-k\rho_k/n$, which corrects this fundamental mistake. (See Anderson, 1994.) So, we must add in η_k to the right of (26).

Then, the correct working down to terms of $O(n^{-1})$ inclusive, continues as:

$$r_k(z)r_h(z) \simeq (\rho_k+\eta_k)(\rho_h+\eta_h) + \sum_{j=1}^{n-1}(\rho_k F_{h,j}+\rho_h F_{k,j})r_j(a)$$

$$- 2\sum_{j=1}^{n-1}\{\rho_j(\rho_k F_{h,j}+\rho_h F_{k,j})+F_{k,j}F_{h,j}\}r_j^2(a).$$

Thus, for $0 < k < n$,

$$E[r_k(z)] \simeq (1 - \frac{k}{n})\rho_k - \frac{2}{n(n+2)}\sum_{j=1}^{n-1}(n-j)\rho_j(\rho_{k+j}+\rho_{k-j}-2\rho_k\rho_j) \qquad (27)$$

which corrects, simplifies and generalises result (2.5) of Davies and Newbold (1980); and, when in addition $0 < h < n$,

$$\mathrm{Cov}[r_k(z), r_h(z)] = E[r_k(z)r_h(z)] - E[r_k(z)]E[r_h(z)]$$

$$\simeq \frac{1}{n(n+2)} \sum_{j=1}^{n-1} (n-j)(\rho_{k+j}+\rho_{k-j}-2\rho_k\rho_j)(\rho_{h+j}+\rho_{h-j}-2\rho_h\rho_j). \qquad (28)$$

This last relation represents an improvement on Bartlett's famous approximation (Bartlett, 1946). We also note that the seven line equation (2.74) of Davies (1977) for $E[r_k(z)r_h(z)]$, from an MA(q), has some algebraic errors and, as a result, is unnecessarily complicated. However, even for stationary ARMA(p, q) processes, as explained in Anderson and De Gooijer (1988), we still prefer our original type of approximation to that of this postcript - but, perhaps, with the expansion, giving (9), taken rather further then - as, computationally, we can use the simpler, raw, matrix-trace expressions for the moments (instead of their complicated explicit evaluations).

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APPENDIX

We give a full comparison, here, of Hasza's approximations to the

Table 3. Comparison of the approximations to the means and standard deviations for the serial correlations at the first four lags for series of length 50, from our finite series considerations and from Hasza's asymptotic theory, with Hasza's simulation results (obtained from 1000 replications) and exact values from numerical integration, given process models of the form $(1 - B)Z_t = (1 - \theta B)A_t$ and various θ.

θ	lag k	1	2	3	4		1	2	3	4	θ
-.8	Asymptotic Mean	.91	.76	.61	.47	Asymptotic SD	.05	.14	.23	.31	-.8
	Monte Carlo Mean	.91	.77	.65	.54	Monte Carlo SE	.05	.12	.17	.20	
	Finite Series Mean	.91	.77	.65	.54	Finite Series SD	.04	.09	.13	.16	
	Exact Mean	.91	.77	.66	.55	Exact SD	.05	.12	.16	.20	
-.4	Asymptotic Mean	.90	.75	.60	.46	Asymptotic SD	.06	.14	.23	.32	-.4
	Monte Carlo Mean	.90	.76	.64	.54	Monte Carlo SE	.05	.12	.17	.20	
	Finite Series Mean	.90	.76	.64	.53	Finite Series SD	.04	.09	.13	.16	
	Exact Mean	.90	.77	.65	.55	Exact SD	.05	.12	.17	.20	
0	Asymptotic Mean	.85	.71	.56	.41	Asymptotic SD	.09	.18	.26	.35	0
	Monte Carlo Mean	.86	.73	.62	.52	Monte Carlo SE	.08	.14	.18	.20	
	Finite Series Mean	.86	.73	.61	.51	Finite Series SD	.06	.10	.14	.17	
	Exact Mean	.86	.74	.63	.53	Exact SD	.08	.13	.17	.20	
+.4	Asymptotic Mean	.62	.47	.32	.18	Asymptotic SD	.27	.36	.44	.53	+.4
	Monte Carlo Mean	.71	.61	.52	.44	Monte Carlo SE	.16	.18	.20	.21	
	Finite Series Mean	.69	.59	.50	.41	Finite Series SD	.14	.17	.19	.21	
	Exact Mean	.71	.62	.53	.45	Exact SD	.16	.18	.20	.21	
+.8	Asymptotic Mean	-3.41	-3.55	-3.70	-3.85	Asymptotic SD	3.42	3.50	3.58	3.67	+.8
	Monte Carlo Mean	.21	.19	.16	.14	Monte Carlo SE	.21	.20	.19	.18	
	Finite Series Mean	.20	.17	.15	.13	Finite Series SD	.23	.23	.22	.21	
	Exact Mean	.22	.19	.17	.14	Exact SD	.20	.19	.18	.18	

means and standard deviations for the serial correlations at the first four lags, for length-50 realisations of the various IMA(1,1) models that he considered, with the corresponding approximations achieved from our finite-series methods. See Table 3. As arbitrator, we use the values which Hasza himself obtained by simulating 1000 replications of each model, and Table 3 also compares these simulated results with exact ones derived from our finite-series considerations and numerical integration. (Note that, in virtually every case, the finite series approximation outperforms the asymptotic one - sometimes dramatically. In the four cases, out of forty, where the asymptotic result shows "better", the two approximations are on opposite sides of the simulated result; and then the difference in closeness of only .01 each time could well be ascribed to rounding error.)

For higher lags (not reported by Hasza), or for values of θ closer to 1, the disparity between the asymptotic and finite-series approximations is greater, with the Hasza approximations deteriorating rapidly with increasing k/n or increasing θ. This is unlike our finite-series formulae, which would be close to values achieved by simulation for all k. However, to be fair, Hasza also reported similar results for series of length 250 from the same models; when, apart from the case $\theta = +.8$, his approximations are not so markedly inferior to ours (on restricting attention to just these initial four lags which, relative to the much longer series length, are indeed now very low lags).

Of course, with θ extended to +1, model (4) reduces to white noise; that is, the series realisations are then merely random samples of n standardised normal variates - and, in such a situation, one would expect a sample size, even of only 50, to allow confident use of appropriate asymptotic formulae. But now the Hasza results (as he indeed noted) are completely inadequate, as they are for all θ moderately close to 1 (where what value of θ might be termed just "moderately close to 1" is a monotonic increasing function of series length, n). However, our finite-series results ensure good approximations are obtained, no matter how short the series are, and no matter how close θ is to +1. In fact, our results become exact when θ actually attains the value 1.

Then, for instance, (12) yields

$$E[c_k^{(n)}] = (n-k)(n\delta_k - 1)\sigma^2 n^{-2} \qquad (0 \leq k < n)$$

$$= \begin{cases} \{(n-1)/n\}\sigma^2 & (k = 0) \\ -\{(n-k)/n^2\}\sigma^2 & (0 < k < n); \end{cases}$$

so (7) gives

$$E_{k,1}^{(n)} = -\frac{(n-k)}{n(n-1)} \qquad (0 < k < n),$$

which agrees exactly with Moran's well-known result for $E[r_k^{(n)}]$, given a white noise process (Moran, 1948).

Finally, we note that numerical integration (as described in De Gooijer, 1980, allows us to get "exact" values for the mean and standard deviations, from finite series considerations, where "exact" is interpreted as being accurate to any degree of precision that we may chose.

As an indication of why the asymptotic approximations perform so much more poorly than the finite-series theory ones, compare Hasza's and our approximations for $E[r_k^{(n)}]$, $E(n,k,\theta)$ and $E_{k,2}^{(n)}$ respectively, given series realisations of length n from model (4). Hasza's approximation (4.1) for $E[r_k^{(n)}]$, as given in (5) above, is bound to be unsound globally, since then

$$\sum_{k=1}^{n-1} E(n,k,\theta) = -(n-1)[2.6675 + 10.652\theta/\{(1-\theta)^2 n\}]; \qquad (29)$$

whereas, in fact, we have from (25) that

$$\sum_{k=1}^{n-1} E[r_k^{(n)}] \cong -\frac{1}{2}, \qquad (30)$$

for any series whatsoever of any length $n > 1$. [That is, (30) is an inevitable algebraic constraint imposed by the "mean-correcting" in the definition of $r_k^{(n)}$.]

For instance, given a random walk (ie $\theta = 0$) of length 50, the average bias of (29) per lag ($0 < k < n$) is the staggering -2.6675; while our two finite series approximations are virtualy without global bias. $E_{k,2}^{(n)}$, our second finite-series approximation for $E[r_k^{(n)}]$ is complicated to write out in terms of powers in k and n, so we just demonstrate our point with the first (less accurate) finite-series approximation, which for a random walk is, from (12),

$$E_{k,1}^{(n)} = 1 - 5k/n + (6k^2 - 1)/n^2 - (2k^3 - k)/n^3 .$$

Then

$$\sum_{k=1}^{n-1} E_{k,1}^{(n)} = -\frac{1}{2} + \frac{1}{2n^2} ,$$

which, when n = 50, gives an average bias of less than +.0000041 per lag.

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BOOK REVIEW

Title : Aide Multicritère à la Décision : Méthodes et Cas

Authors : Bernard Roy and Denis Bouyssou

Editor : Economica, Collection Gestion, 1993, 695p.

Reviewer : Bertrand Mareschal

Review :

This book written in French by Bernard Roy and Denis Bouyssou is a comprehensive and user-oriented synthesis of the state of the art in multicriteria decision aid (MCDA). It is oriented towards the user of multicriteria methods and establishes an important link between theory and practice.

The book is structured as follows :

Chapter 1 : Introduction and general concepts

This first chapter gives a summary of the underlying MCDA methodology as it is presented in the previous book of B. Roy (*Méthodologie multicritère d'aide à la décision*, Economica, 1985). Different basic concepts are defined : decision process and decision aid, alternatives, preference modelling, criteria. The problem of the aggregation of preferences is also introduced.

Chapter 2 : Family of criteria : consistency and independence

The choice of an adequate family of criteria is a crucial step in a decision process. The authors define minimal consistency axioms that should reasonably be fulfilled by the criteria. These lead to the definition of the notion of *consistent family of criteria*. Several properties of such families are then studied and advice are given for the selection of the criteria. Different actual cases studies are mentioned as examples.

Chapter 3 : Conflicting criteria and elementary multicriteria aggregation procedures

In this chapter, the authors emphasise the notion of conflicting criteria and its link with the way that pairwise comparisons between actions should be made. The *concordance–discordance* principle is then introduced and leads to the problem of assessing the relative importance of the criteria. More generally, the important role of inter-criteria information is analysed. The central concept of a *multicriteria aggregation procedure* is then introduced : it is defined as a rule that associates to a performance table and additional inter-criteria information a preference relation on the set of alternatives. Several examples of elementary procedures are given (e.g. lexicographic aggregation, concordance–discordance principle, weighted sum). Finally, the problem of taking into account the relative importance of the criteria is more deeply discussed. In particular, the authors insist on the dependence between the aggregation procedure and the meaning of coefficients such as weights associated to the criteria.

Chapter 4 : Multicriteria aggregation procedures based on a unique synthesis criterion

The general principles of aggregation procedures that build a preference relation on the set of alternatives based on a single synthesis criterion are first given. Different types of procedures are then considered, depending on the kind of criterion used : either a true criterion (e.g. additive value function) or a more sophisticated quasi– or pseudo–criterion. A specific section is also devoted to the expected utility theory.

Chapter 5 : Multicriteria aggregation procedures not based on a unique synthesis criterion (e.g. ELECTRE methods)

These procedures define preference relation structures that can encompass incomparability and intransitivity as in the outranking method developed by the European MCDA school. They are compared to those of the previous chapter and the respective advantages of both approaches are explained. More specifically, the following methods are considered : ELECTRE I, IS, II, III, IV, TACTIC and PROMETHEE. The concept of robustness is also introduced and discussed, as well as the properties of such procedures and their possible axiomatization.

Chapter 6 : MCDA based on methods of the ELECTRE type

The ELECTRE type methods belong to the class of procedures described in the previous chapter. Their principles are fully described, according to their purpose : selection problem (ELECTRE I and IS), sorting (ELECTRE TRI) or ranking (ELECTRE II, III and IV ; PROMETHEE I and II) of the alternatives. The corresponding algorithms are presented and useful practical as well as theoretical comments are given.

Chapter 7 : Interactive methods

After analysing the American school in chapter 4 and the European one in chapters 5 and 6, the authors consider interactive MCDA methods. In this case, no aggregation procedure is explicitly used but instead a stepwise dialogue between the decision maker and an analyst allow to progressively issue recommendations. The problem of the convergence of such procedures is analysed and this leads to distinguish two notions : an *algorithmic convergence* and a *psychological convergence.* A general structure for interactive methods is then proposed. Four specific methods are also presented. A section is also devoted to methods such as PREFCALC, that handle ranking or sorting problems (as opposed to the majority of interactive methods that are designed to assist in selection problems).

Chapter 8–9–10 : Case studies

These last chapters describe three case studies :

- selection of a sorting machine (application of ELECTRE IS)

- location of a nuclear power plant (comparison of two approaches : multi–attribute utility model and ELECTRE III)
- programming of heavy investments (comparison between ELECTRE IV and an empirical approach) The different steps of each study are well detailed : definition of the alternatives and of the criteria, selection of an appropriate method, analysis of the results, ...

Three additional sections complete the book :

- a detailed listing of a hundred actual applications of MCDA methods.
- an impressive list of references (about 300 papers).
- a useful index of specific terms pertaining to MCDA.

As a conclusion, this book is an invaluable source of information for anybody involved in or interested by multicriteria decision aid. The theoreticians will find here a summary of the state of the art in the field, as well as a link with more practical aspects. Potential users of MCDA methods will gain a better understanding of how to handle their decision problems efficiently and to avoid common pitfalls. The only drawback of the book is that it is currently accessible to French–reading people only.