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A new fundamental preference structure : treshold order

M. Abbas[1,2] Ph. Vincke[2]

Institut de Mathématiques[1]
Université des Sciences et de la Technologie Houari Boumedienne
BP 32, El-Alia
16111 Alger, Algeria

Service de Mathématiques de la Gestion[2]
Université Libre de Bruxelles
Bld du Triomphe, CP 210/01
1050 Bruxelles, Belgium

Abstract

In this paper we present a new preference structure which may be considered as a fundamental structure in preference modelling. Before characterizing this structure, we investigate thoroughly the threshold graphs class which corresponds to it. In particular, we propose a new characterization of this class of graphs in terms of complete preorder (S_G) on the vertex set of graph G.

Keywords : treshold order, treshold graph, decision theory, preference modelling

1 Introduction

There are several examples of complete preference structures which can be built up by "glueing together" an asymmetric relation P and a symmetric relation I. It is the case of complete preorders (Roubens and Vincke (1985)), semiorders (Roberts (1969) and Pirlot (1990, 1991)) and interval orders (Fishburn (1985)). For instance, the graph of the symmetric part of an interval order is an interval graph (i.e. the intersection graph of a family of intervals of the real line) and the asymmetric part is any transitive orientation of the complementary graph. Starting from the graph of any symmetric relation, it is always possible to transitively orientate the complementary graph provided the original graph is a co-comparability graph. In other words starting from any I whose graph is a co-comparability graph, one can build a P such that {P, I} is a complete preference structure in the sense of Roubens and Vincke (1985), Abbas and Vincke (1993) or Abbas (1994). The co-comparability graphs form a sub-class of the famous and much studied family of perfect graphs (Golumbic (1980), Berge and Chvatal (1984), Zemirline (1987) and Maffray (1992)). In this paper, we provide an additional example of such a construction. We start from a threshold graph as the graph of I and by transitively orienting the complementary we define a preference structure we naturally call threshold order. Threshold graphs are a sub-class of perfect graphs introduced and characterized by Chvatal and Hammer (1977). It is a particularly interesting one as its structure can be described in a very precise manner. In addition, this structure can be interpreted in the framework of classification theory as it is connected with a particular type of hierarchic tree (this aspect will be developed elsewhere). A large quantity of theoretic as well as practical papers have been devoted to threshold graphs and we briefly review a number of them bellow. The general ideas of the different papers are about characterization, extension, polyhedral study, nonthresholdness of a graph and the use of such graphs to modelize concrete problems. First, we recall that a graph G = (V,E) is a threshold graph if and only if there exists a threshold $t \in \mathbb{N}_0$ and a mapping $w : V \rightarrow \mathbb{N}_0$ such that $\forall$ $S \subseteq V$, S is a stable set if and only if $\sum_{v \in S} w(v) \leq t$, where a set $S \subseteq V$ is a stable set of graph G = (V,E) if and only if $\forall$ v, u $\in$ S, $\{v,u\} \notin E$. Numerous characterizations of threshold graphs were discovered. We cite hereafter those which are at the origin of this work. The readers interested by other characterizations, under a different aspect, are referred to Chvatal and Hammer (1977), Henderson and Zalcstein (1977), Benzaken and Hammer (1978), Golumbic (1980), Hammer et al. (1981), Cogis (1982a).

In their paper, Benzaken and Hammer (1978) introduced an extension of threshold graphs. These graphs, called domishold graphs, have the property that there exists a threshold $t \in \mathbb{N}_0$ and a mapping $w : V \rightarrow \mathbb{N}_0$ such that $\forall\ D \subseteq V$, D is a dominating set if and only if $\sum_{v \in D} w(v) \geq t$, where a set $D \subseteq V$ is a dominating set if and only if $\forall\ v \in V \backslash D$, $\exists\ a \in V$ such that $\{v,a\} \in E$. Payan (1980) investigated two classes of graphs called equistable and equidominating graphs, obtained, respectively, from threshold and domishold graphs, by replacing the inequalities by equalities. Mahjoub (1981), characterized the polytope P(G) corresponding to the class of domishold graphs. At the same moment, Hammer et al. (1981), after defining a threshold sequence, as being a sequence of n non negative integers corresponding to the sequence of degrees of a threshold graph, they proposed numerous characterizations, in particular, the set of threshold sequences at minimum distance from an arbitrary degree sequence. Peled and Srinivasan (1989) studied the polytope of degree sequence. The notion of threshold dimension t(G) of a graph G was also introduced by Chvatal and Hammer (1977), as being the smallest number t such that there are threshold graphs G_i, $1 \leq i \leq t$, whose edge union is G. Usually, the notion of dimension is regarded as an intersection, we can prove that the threshold dimension t(G) of an arbitrary graph G can be defined in an alternate but equivalent manner using intersection threshold graphs. Chvatal and Hammer (1977) showed that finding t(G) is NP-hard. Yannakakis (1982) showed that it is NP-complete to determine if a given graph has a threshold dimension at most 3. Two years later, Cozzens and Leibowitz (1984) showed that determining if a graph has threshold dimension 2 is also NP-complete. Different sufficient conditions for a graph to have threshold dimension 2 are obtained by Ibaraki and Peled (1981). Graphs having threshold dimension 2 are called 2-threshold graphs. These graphs have been studied by Hammer et al. (1989). Hammer and Mahadev (1985), introduced the bithreshold graphs, i.e. graphs G which are the edge-intersection of two threshold graphs G_1 and G_2 defined on the same vertex set, with the property that every stable set of G is also a stable set of G_1 or G_2. They proposed a polynomial recognition and decomposition algorithm for bithreshold graphs. A signed balanced graphs defined by threshold conditions was introduced by Benzaken et al. (1985), generalizing threshold graphs. In fact, they are equivalent to graphs with Dilworth number at most two. An important case of this class (difference graphs) was recently investigated by Hammer et al. (1990).

In a paper, Cogis (1982a) deduces a set of known characterizations of Ferrers digraphs by investigating the connection between symmetric Ferrers digraphs and threshold graphs. In fact, the author provides a positive answer to this question through the usual connection : a graph is but a symmetric digraph (that is if (x,y) is a directed edge of the digraph, then (y,x) is too) without loops. Ferrers digraphs were defined by

Riguet (1951). A Ferrers digraph is a digraph $G = (X,U)$ such that GG^dG is a partial digraph of G, where G^d is the dual graph of G, i.e. $G^d = (V,U^d)$, and $(a,b) \in U^d$ if and only if $(b,a) \notin U$. Bouchet (1971), was able to extend the Dushnik and Miller's dimension of a partial order (1941) to any digraph. On the one hand, Cogis (1982b) showed that the problem of finding the Ferrers dimension of a digraph and the problem of finding the order dimension of a partial order are polynomially equivalent (in the sense of Karp (1972)). On the other hand, he showed (Cogis (1980)) that the problem of finding the Ferrers dimension of a digraph is polynomially equivalent to the problem of finding the threshold dimension of a split graph.

In the context of measuring how "nonthreshold" a given graph is, many measurements beside the threshold dimension have been studied. Hammer et al. (1981) introduced the threshold gap; while Peled and Simeone (1987) introduced the threshold measure, and more recently, Wang and Williams (1991), introduced the threshold weight of a graph G. The latter consists, in contrast to other approaches to thresholdness which assign weights only to the vertices, to assign weights to both vertices and edges. Formally, given any graph $G = (V,E)$, we assign a positive threshold value t to the graph, non negative weights $w_i \geq 0$ to each vertex $i \in V$ and non negative weights $w_e \geq 0$ to each edge $e \in E$, such that for any subset $S \subseteq V$, S is a stable set of G if and only if the total weight of the vertices and of the edges over the subgraph G(S) induced by S is less than the threshold value, i.e. $\sum_{e \in E(S)} w_e + \sum_{i \in S} w_i \leq t-1$ if and only if S is a stable set.

The threshold weight W(G) is defined as the optimal value of the following linear programming problem :

$$\min \sum_{e \in E} w_e$$

$\sum_{i \in S} w_i \leq t-1$, for each maximal stable set $S \subseteq V$;

$w_i + w_j + w_e \geq t$, for each $e = \{i,j\} \in E$;

$w \geq 0$.

The optimal value W(G) equals 0 if and only if G is a threshold graph. Therefore, when the graph G is not a threshold graph, W(G) may be interpreted as a measure of the nonthresholdness of G. Hammer et al. (1991) studied the structure of the networks in which connectedness and disconnectedness of the networks can be expressed by a threshold system. This means that the elements of the network have a certain "destruction cost" and that the enemy can disconnect the network if and only if they pay a large enough price. In their paper, they give polynomial algorithms for the recognition of such networks, and for the determination of the appropriate costs and threshold value. Finally, an interesting result given by Leibowitz (1978) is that every threshold graph has an interval representation using intervals of at most two different lengths.

This review shows that threshold graphs have received extensive attention. The fact that the structure has been used to define a threshold-dimension shows that it is in some sense fundamental. The status of the threshold order structure is in many respects similar to the one of interval orders and semiorders. They can all be built up starting from a symmetric relation by transitively orienting the complementary and they are all associated with a notion of dimension (resp. interval order, semiorder, threshold order dimension). That's why we promote the threshold order to the dignity of a fundamental preference structure.

In the present paper, we try to characterize a preference structure which corresponds to the class of threshold graphs. Before characterizing this structure, we study thoroughly the threshold graphs class which corresponds to it. In particular, we propose a new characterization of this class by a certain complete preorder (S_G) on the vertex set of graph G. Before introducing some necessary definitions and results, we suppose that all the considered sets are finite and non empty. For the multicriteria decision-aid terminology, we essentially refer to Vincke (1992), whereas for the graph theory terminology, the reader is referred to Berge (1973).

2 Preliminary definitions and results

Let us consider a set V of alternatives (objects, actions, decisions, candidates,...). We assume that a subject who must compare two alternatives a and b, is able to express his preference (P) between the two alternatives or his indifference (I) between the two, i.e. incomparability (J) is not permitted in this approach. This situation is often met in the preference modelling literature (cf. Roubens and Vincke (1983, 1985), Roy (1980, 1985), Roy and Bouyssou (1993) and Vincke (1992)), and it can be represented as follows :

RELATIONAL REPRESENTATION
aPb (resp. bPa) if a is preferred to b (resp. b is preferred to a);
aIb if a and b are indifferent.

GRAPHIC REPRESENTATION
an arc from a to b (resp. b to a) if a is preferred to b (resp. b is preferred to a);
an edge between a and b if a and b are indifferent.

Definition 1. *A preference structure without incomparability is a pair* {P, I} of *binary relations defined on the same set* V, *such that :* $\forall$ a, b $\in$ V :

(i) aPb $\Rightarrow$ b(notP)a *(*P *is asymmetric);*

(ii) aIa *(*I *is reflexive);*

(iii) aIb $\Rightarrow$ bIa *(*I *is symmetric);*

(iv) a(notP)b and b(notP)a $\Leftrightarrow$ aIb.

Definition 2. *An interval order* {P, I} *on the set* V *is a preference structure without incomparability, such that* PIP $\subseteq$ P.

It is easy to show the following result :

Proposition 3. *Let* P *and* I *be two binary relations defined on the same non empty set* V. {P, I} *is an interval order if and only if* {P, I} *is a preference structure which has no* A_i, $1 \leq i \leq 3$ *as an induced subgraph (see figure 1).*

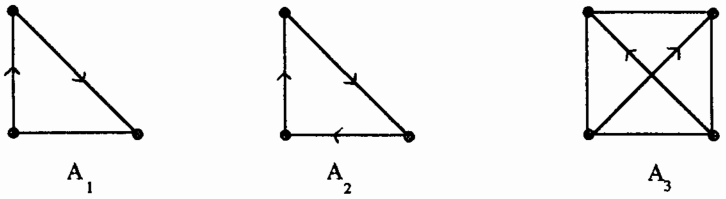

Fig. 1. Interval order forbidden minimal configurations

Definition 4. *A graph* G = (V,E) *is an interval graph if there exists a family of intervals* $\mathfrak{I} = \{I_a : a \in V\}$ *on the real line such that* {a,b} $\in$ E *if and only if* $I_a \cap I_b \neq \varnothing$.

The following results (see Fishburn (1985)) show the equivalence between interval orders and the class of interval graphs.

Theorem 5. *Let* I *be a binary, reflexive, symmetric relation defined on the set* V *and* $\overset{\circ}{I} = I \setminus \{\text{loops}\}$. G = (V, $\overset{\circ}{I}$) *is an interval graph if and only if there exists an interval order* {P, I} *on* V *such that* $I = (P^c)^s$. {P, I} *is an interval order on* V *if and only if* P *is transitive,* G = (V, $\overset{\circ}{I}$) *is an interval graph and* $I = (P^c)^s$.

The following characterization due to Chvatal and Hammer, permits us, in section 4, to define the threshold order structure.

Theorem 6. (Chvatal and Hammer, 1977) : *A necessary and sufficient condition for* G *to be a threshold graph is that* G *does not have* $2K_2$, P_4, *or* C_4 *as an induced subgraph (see figure 2).*

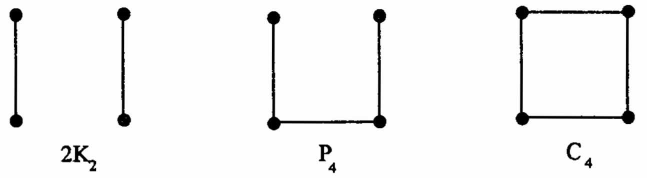

Fig. 2. Threshold graph forbidden minimal configurations

It is easy to prove that :

Proposition 7. *A threshold graph is an interval graph.*

3 Complete preorders and threshold graphs

By analogy with the characterization of domishold graphs by a certain complete preorder on the vertices and by using the dominating sets of a graph (Benzaken, Hammer (1978)). we prove in this section that it is possible to characterize threshold graphs by a certain complete preorder on the vertices and by using the stable sets of the graph.

Let $G = (V,E)$ be a simple graph and let us define a binary relation S_G on V by putting :

xS_Gy $(x, y \in V)$ if and only if $\forall$ S stable set of G such that $x \notin S$ and $y \in S$, then $(S\backslash\{y\})\cup\{x\}$ is a stable set of G.

We shall say that "y may be replaced by x".

Lemma 8. S_G *constitutes a preorder on* V *(reflexive and transitive relation).*

Proof. Reflexivity is obvious. For transitivity, let us suppose we have $x, y, z \in V$ (2 by 2 distinct) such that xS_Gy and yS_Gz and let S be a stable set of G, containing z but not x. If $y \notin S$, then $(S\backslash\{z\})\cup\{y\}$ is a stable set of G (because yS_Gz) containing y but not x. So $T^* = ([(S\backslash\{z\})\cup\{y\}]\backslash\{y\})\cup\{x\}$ is a stable set of G (because xS_Gy). But $T^* = (S\backslash\{z\})\cup\{x\}$. If $y \in S$, then $(S\backslash\{y\})\cup\{x\}$ is a stable set of G (because xS_Gy) containing z but not y. So $T^* = ([(S\backslash\{y\})\cup\{x\}]\backslash\{z\})\cup\{y\}$ is a stable set of G (because yS_Gz). But $T^* = (S\backslash\{z\})\cup\{x\}$. In both cases xS_Gz. □

The main result of this section is the following :

Theorem 9. *Let* G = (V,E) *be a simple graph. The following assertions are equivalent and characterize a threshold graph :*
(i) G *has no induced subgraph isomorphic to* $2K_2$, P_4 *or* C_4.
(ii) *The preorder* S_G *on* V *is complete.*

We prove the result in several steps.

Proposition 10. *If* G = (V,E) *is a threshold graph, then* S_G *is a complete preorder.*

Proof. Indeed, suppose G = (V,E) is a threshold graph and the preorder S_G is not complete. Therefore there exists x, y ∈ V such that (x,y) and (y,x) ∉ S_G, i.e. ∃ S_1 (resp. S_2) a stable set of G such that x ∉ S_1 (resp. y ∉ S_2), y ∈ S_1 (resp. x ∈ S_2) and $(S_1\setminus\{y\})\cup\{x\}$ (resp.$(S_2\setminus\{x\})\cup\{y\}$ is not a stable set of G.

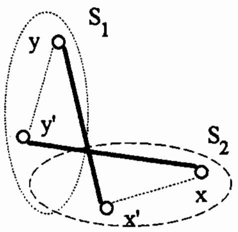

Fig. 3.

This implies that there exists y' ∈ S_1 (resp. x' ∈ S_2) such that {x,y'} (resp. {y,x'}) ∈ E. Furthermore, {y,y'}(resp.{x,x'}) ∉ E because S_1 (resp.S_2) is a stable set of G (see figure 3). So the subgraph induced by x, x', y and y' is isomorphic to $2K_2$, P_4 or C_4, which is impossible. ☐

Remark. It is easy to see that none of the graphs $2K_2$, P_4 or C_4 have a complete preorder with respect to S_G.

A vertex m of G = (V,E) is called minimal if it is minimal with respect to S_G, i.e. ∀ x ∈ V, xS_Gm. " m may be replaced by any vertex of G ".
It is easy to prove the following properties :

Proposition 11. *1) Any dominating vertex is minimal.*
2) If a graph has a dominating vertex, then every minimal vertex is dominating.

Lemma 12. *Let* $G = (V,E)$ *be a graph such that the corresponding preorder* S_G *is complete and let* m *be a minimal vertex of* G. *If* m *is neither an isolated nor a dominating vertex, then every* $x \notin N(m) \cup \{m\}$ *is an isolated vertex of* G.

Proof. Let S be a maximal stable set of G containing m and let y be a vertex of $V \backslash S$. We have yS_Gm because m is a minimal vertex of G. So $(S\backslash\{m\})\cup\{y\}$ is a stable set of G. This implies that y is not adjacent to any vertex of $(S\backslash\{m\})$. Furthermore, y is adjacent to m (because S is maximal). But y is an arbitrary vertex of $V\backslash S$, which implies that $(S\backslash\{m\})$ is an isolated vertex set of G, i.e. $N(m) = V\backslash S$. □

Lemma 13. *If* S_G *is complete,* m *is a minimal vertex of* $G = (V,E)$ and G_m *the subgraph induced by* $V\backslash\{m\}$, *then the preorder* S_{G_m} *is also complete.*

Proof. Let $x, y \in V\backslash\{m\}$ and assume xS_Gy. Let S be a stable set of G_m containing y but not x. Assume first that m is isolated in G, then $S\cup\{m\}$ is a stable set of G containing y but not x. Hence $[(S\cup\{m\})]\backslash\{y\}]\cup\{x\}$ is a stable set of G and by deleting m, $(S\backslash\{y\})\cup\{x\}$ is a stable set of G_m, showing that $xS_{G_m}y$. If m is a dominating vertex of G, then S is a stable set of G and $(S\backslash\{y\})\cup\{x\}$ is a stable set of G_m, showing that $xS_{G_m}y$. Finally, let us consider the case where m is neither isolated nor dominating in G. If $y \notin N(m)$ (resp. $x \notin N(m)$), then $yS_{G_m}x$ (resp. $xS_{G_m}y$), because by lemma 12, y (resp. x) is an isolated vertex in G and therefore in G_m. If $x, y \in N(m)$, then S is a stable set of G because S is a stable set of G_m containing y but not x. Furthermore, $(S\backslash\{y\})\cup\{x\}$ is a stable of G, does not contain m, because xS_Gy. So $(S\backslash\{y\})\cup\{x\}$ is a stable set of G_m, showing that $xS_{G_m}y$. In each case, we obtain that S_{G_m} is a complete preorder. □

Proposition 14. *A graph* $G = (V,E)$ *having the property that the preorder* S_G *is complete, cannot have any induced subgraph isomorphic to* $2K_2$, P_4 or C_4.

Proof. Assume that G with complete preorder S_G has an induced subgraph H isomorphic to $2K_2$, P_4 or C_4. By removing a minimal vertex $m \notin H$ (if possible) and continuing this process as many times as possible, we shall eventually arrive (by lemma 13) to a graph G' (with complete preorder) having a minimal vertex m in its induced subgraph H. We can find two vertices $x, y \in V\backslash\{m\}$ such that $x \notin N(m)$ (or $y \notin N(m)$). By lemma 12, m is not minimal, a contradiction. □

So theorem 9 is proved.

Proposition 15. *If* $G = (V,E)$ *is a simple graph, then we have* $S_{G^c} = \bar{S}_G$, *where* G^c *is the complementary graph of* G and $\bar{S}_G$ *the converse of preorder* S_G.

Proof. Indeed, if $S_{G^c} \neq \bar{S}_G$, then there exists $x, y \in V$ such that $(x,y) \in S_{G^c}$ and $(x,y) \notin \bar{S}_G$ or there exists $x', y' \in V$ such that $(x',y') \notin S_{G^c}$ and $(x',y') \in \bar{S}_G$. Assume first $(x,y) \in S_{G^c}$ and $(y,x) \notin S_G$, which implies for every stable set S of G^c such that $x \notin S$, $y \in S$, then $(S\backslash\{y\})\cup\{x\}$ is a stable set of G^c; furthermore, there exists a stable set S' of G such that $y \notin S'$, $x \in S'$ and $(S'\backslash\{x\})\cup\{y\}$ is not a stable set of G. $S = \{y,x'\}$ is a stable of G^c, containing y but not x. But $(S\backslash\{y\})\cup\{x\} = \{x,x'\}$ is not a stable set of G^c, a contradiction. If there exists $(x',y') \notin S_{G^c}$ and $(y',x') \in S_G$, then by similar arguments as in the first case, we obtain a contradiction. □

Other binary relations have been studied, for example :

- The binary relation D_G defined on V by :

xD_Gy iff $d_G(x) \leq d_G(y)$, where $d_G(x)$ is the degree of vertex x of G.

It is obvious that relation D_G constitutes a complete preorder on V.

- The binary relation N_G defined on V by :

xN_Gy iff $N_G(x) \subseteq N_G(y) \cup \{y\}$, where $N_G(x)$ is the set of vertices adjacent to x in G.

It is also well-known that relation N_G constitutes a preorder on V, called "vicinal preorder".

Proposition 16. *Consider* $G = (V,E)$ *to be a simple graph. We have :*

(i) $S_G = N_G$;

(ii) $S_G \subseteq D_G$.

Furthermore, G *is a threshold graph if and only if* $S_G = D_G$.

Proof. • If (i) is not true, then there exist $x, y \in V$ such that $(x,y) \in S_G\backslash N_G$ or there exist $x', y' \in V$ such that $(x',y') \in N_G \backslash S_G$. Assume first, $(x,y) \in S_G$ and $(x,y) \notin N_G$ which implies that for any stable set S of G such that $x \notin S$ and $y \in S$, then $(S\backslash\{y\})\cup\{x\}$ is a stable set of G; furthermore $N_G(x) \not\subset N_G(y) \cup\{y\}$. The latter assertion implies that there exists $x' \in N_G(x) \setminus N_G(y) \cup \{y\}$. The set $S = \{y,x'\}$ is a stable set of G containing y but not x. But $(S\backslash\{y\})\cup\{x\} = \{x,x'\}$ is not a stable set, which is impossible. If $(x,y) \notin S_G$ and $(x,y) \in N_G$, then there exists a stable set S such that $x \notin S$, $y \in S$ and $(S\backslash\{y\})\cup\{x\}$ is not a stable set of G; furthermore, $N_G(x) \subseteq N_G(y)\cup\{y\}$, a contradiction.

• (ii) obvious.

• Finally, by (ii) and theorem 9, it is sufficient to show that if G is a threshold graph, then $D_G \subseteq S_G$. Suppose there exist $x, y \in V$ such that $(x,y) \notin S_G$ and $(x,y) \in D_G$, then there exists a stable set S' of G such that $x \notin S'$, $y \in S'$ and $(S\backslash\{y\})\cup\{x\}$ is not a stable of G; furthermore $d_G(x) \leq d_G(y)$.

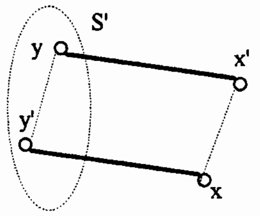

Fig. 4.

The latter assertion implies that there exists $x' \in N_G(y)\backslash N_G(x)$, but x, x', y, and y' induced a subgraph isomorphic to $2K_2$, P_4 or C_4 (see figure 4). So G is not a threshold graph. □

4 Threshold graph and Preference structure

Definition 17. *Let* P *and* I be *two binary relations defined on the same set* V. *A preference structure* {P, I} *on the set* V *is a threshold order if and only if :*

(i) $P^2 \subseteq P$ *(i.e.* P *is transitive);*

(ii) $\forall$ a, b, c, d $\in$ V *(*2 *by* 2 *distinct), we have* : aIb, bPc, cId $\Rightarrow$ aId.

It is easy to show the following result :

Proposition 18. *Let* P *and* I be *two binary relations defined on the same set* V. {P, I} *is a threshold order on set* V *if and only if* {P, I} *is a preference structure which has no* A_i, $1 \leq i \leq 5$ *as an induced subgraph (see figure 5).*

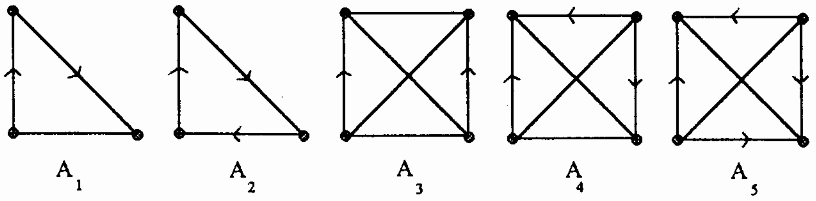

Fig. 5. Threshold order forbidden minimal configurations

Corollary 19. *A threshold order is an interval order structure without configurations* A_4 *and* A_5.

We establish now the equivalence between threshold graphs and threshold orders.

Theorem 20. *Let* I *be a binary symmetric relation on the set* V. *There is a binary relation* P *on the set* V, *such that* {P, I} *is a threshold order on the set* V *if and only if* (V, $\overset{\circ}{I}$) *is a threshold graph, where* $\overset{\circ}{I}$ = I \{loops}.

Proof. Necessary condition. By proposition 7 and theorem 5, there exists a binary asymmetric relation P, such that {P, I} is an interval order structure. Therefore, it is sufficient to show, by contradiction, the second condition of definition 17. Let us suppose P is a transitive orientation of the complementary graph of graph (V,I), and let a, b, c, d ∈ V (2 by 2 distinct), such that aIb, bPc, cId and aPd. We can show that the vertices a, b, c and d induce a configuration of (V, $\overset{\circ}{I}$) isomorphic to $2K_2$, P_4 or C_4, because P is transitive and P ∪ I strongly complete, which is impossible.

Sufficient condition. By proposition 18, it is sufficient to show that {P, I}does not have an induced subgraph isomorphic to the minimal configurations A_i, i = 1,...,5 (see figure 5) implies (V, $\overset{\circ}{I}$) is a threshold graph. Indeed, if graph (V, $\overset{\circ}{I}$) is not a threshold graph, there exists an induced subgraph of (V, $\overset{\circ}{I}$) isomorphic to $2K_2$, P_4 or C_4 (see figure 2). We can suppose, without loss of generality, that a, b, c, d ∈ V, are the vertices of $2K_2$, (resp. P_4 ; C_4) and aPc (see figure 6). Therefore, we obtain aPd, bPc and bPd (resp. bPc, bPd; bPd), because P is transitive.

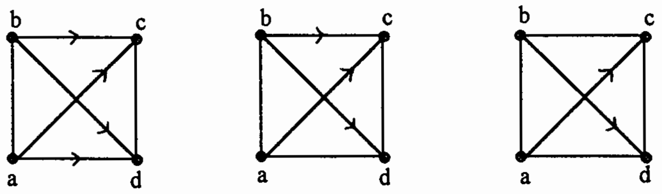

Fig. 6. □

5 Potential application

On the one hand, Monjardet and Jacquet-Lagreze (1978) wished to see the development of some work on the problem of finding a Guttman scale at a minimum distance from any given correspondence. On the other hand, we give here an example from Leibowitz (1978) on the link between the Guttman scale and a threshold graph. This suggests that the former problem could be reformulated as finding a threshold order at minimal distance from a given correspondence. As the threshold graphs class has been extensively studied, this simple remark could perhaps help finding an efficient algorithm.

Link between Guttman scale and threshold order. Let X be a set of propositions and let Y be a set of subjects in a psychological experiment. A subject either agrees or disagrees with a proposition . A Guttman scale is a linear ordering of $X \cup Y$, such that a subject agrees with all items following it and disagrees with all items preceding it. Let G be an undirected graph with vertex set $X \cup Y$ constructed as follows : X forms a stable set; Y forms a clique; subject y is adjacent to proposition x if and only if subject y agrees with proposition x. We have the equivalence : there exists a Guttman scale for $X \cup Y$ if and only if G is a threshold graph.

We can now add that the linear ordering of the Guttman scale induces a transitive orientation of the complementary graph of G which makes it a threshold order.

5 Conclusion

Several concrete problems require to be modelized by semi-oriented graphs since the modelling by non oriented graphs is not sufficient. But since various theoretic works as well as practical works are devoted to threshold graphs, we have introduced the threshold order structure in order to enrich the set of models which can be used in decision aid.

6 Open questions

1) The following characterization is due to Henderson and Zalcstein (1977) :
A graph $G = (V,E)$ is a threshold graph if and only if :
$\exists\, c : V \rightarrow \mathbb{N}$, $t > 0$ such that $\forall\, a, b \in V$, $a \neq b : \{a,b\} \in E \Leftrightarrow c(a) + c(b) > t$.
Find a numerical representation of a threshold order.

2) Characterize the preorders corresponding to the degree relation.

3) Characterize the preorders corresponding to domishold graphs which are defined below.
Let us consider $G = (V,E)$ to be a simple graph. We define the binary relation δ_G on set V : $a \delta_G b$ if and only if $\forall$ dominating set D of G such that $a \notin D$ and $b \in D$, then $(S \setminus \{b\}) \cup \{a\}$ is a dominating set of G. Benzaken and Hammer (1978) showed that : G is a domishold graph if and only if δ_G is a complete preorder.

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7 References

Abbas, M. and Vincke, Ph. Preference structures and threshold models. *Journal of Multi-Criteria Decision Analysis*. Vol. 2, (1993) 171-178.

Abbas, M. Any complete preference structure without circuit admit an interval representation. *Theory and Decision*. To appear. (1994).

Benzaken, C. and Hammer, P.L. Linear separation of dominating sets in graphs, *Annals of Discrete Mathematics* 3 (1978) 1-10.

Benzaken, C., Hammer, P.L. and De Werra, D. Threshold characterization of graphs with Dilworth number two. *Journal of Graph Theory* 9 (1985) 245-267.

Berge, C. and Chvatal, V. Topics on Perfect Graphs. *Annals of Discrete Mathematics*. (1984).

Berge, C. *Graphs and Hypergraphs*. North- Holland. Amsterdam. (1973).

Bouchet, A. *Etude combinatoire des ordonnés finis - Applications*. Thèse de Doctorat d'Etat, Université de Grenoble (1971).

Chvatal, V. and Hammer, P.L. Aggregation of inequalities in integer programming. *Annals of Discrete Mathematics*. 1 (1977) 145-162.

Cogis, O. *Dimension de Ferrers des graphes orientés*. Thèse de Doctorat d'Etat. Université Pierre et Marie Curie. Paris. (1980).

Cogis, O. Ferrers digraphs and threshold graphs. *Discrete Mathematics*. 38 (1982a) 33-46.

Cogis, O. On the Ferrers dimension of a digraph. *Discrete Mathematics*. 38 (1982b) 47-52.

Cozzens, M.B. and Leibowitz, R. Threshold dimension of graphs, *SIAM J. Algebraic Discrete Methods*. 5 (1984) 579-594.

Dushnick, B. and Miller, E.W. Partially ordered sets. *Amer. J. Math.* 63, 600-610. (1941).

Fishburn, P.C. *Interval orders and interval graphs*. Wiley, New-York. (1985).

Golumbic, M.C. *Algorithmic Graph Theory and Perfect Graphs*. Academic Press, New York (1980).

Hammer, P.L. and Mahadev, N.V.R. Bithreshold graphs, *SIAM J. Algebraic Appli. Math.* 6 (1985) 497-506.

Hammer, P.L. Ibaraki, T. and Simeone, B. Threshold sequences, *SIAM J. Algebraic*

Discrete Methods. 2 (1981) 39-49.

Hammer, P.L., Maffray, F. and Queyranne, M. Cut-threshold graphs. *Discrete Applied Mathematics.* 30 (1991) 163-179.

Hammer, P.L., Mahadev, N.V.R. and Peled, U.N. Some properties of 2-threshold graphs. *Networks* 19 (1989) 17-23.

Hammer, P.L., Peled, U.N. and Sun, X. Difference graphs. *Discrete Applied Mathematics.* 28 (1990) 35-44.

Henderson, P. B. and Zalcstein, Y. A graph-theoretic characterization of the pv_{chunk} class of synchronizing primitives. *SIAM. J. COMPUT.* Vol.6. No.1(1977) 88-108.

Ibaraki, T and Peled, U.N. Sufficient conditions for graphs to have threshold number 2, in: P.Hansen, ed., *Studies on Graphs and Disc. Prog.* (North-Holland, Amsterdam, 1981) 241-268.

Karp, R.M. Reducibility among combinatorial problems, in: R.E. Miller and J.W. Thatcher, eds.*Complexity of computers computations* (plenum press, New York, 1972) 85-104.

Leibowitz, R. *Interval counts and threshold numbers of graphs.* Thesis. Rutgers University, (1978).

Maffray, F. *Une étude structurelle des graphes parfaits.* Thèse de Doctorat. Université de Grenoble. (1992).

Mahjoub, A. R. *Polytope des absorbants d'un graphe absorbant à seuils.* Thèse de Doctorat. Université de Grenoble. (1981).

Monjardet, B. et Jacquet-Lagreze, E. Modélisation des préférences et quasi-ordres. *Math. Sci. Hum.* 62 (1978) 5-10.

Payan, C. A class of threshold and domishold graphs : Equistable and equidominating graphs. *Discrete Mathematics.* 29 (1980) 47-52.

Peled, U. and Simeone, B. The threshold measure of a graph. *Annals of the New York Academy of Sciences.* (1987) vol. 476. 441-457.

Peled, U. and Srinivasan, M.K. The polytope of degree sequences. *Lin. Alg. and its Appl.* (1989) 114-115.

Pirlot, M. Minimal Representation of a Semiorder. *Theory and Decision* 28, (1990) 109-141.

Pirlot, M. Synthetic description of a semiorder. *Discrete Applied Mathematics.* 31

(1991) 299-308.

Riguet, J. Les relations de Ferrers. *C.R.Acad.Sci.Paris.* 232 (1951) 1729-1730.

Roberts, F.S. Indifference Graphs, in *Proof Techniques in Graph Theory.* (1969).

Roubens, M. and Vincke, Ph. Linear orders and semiorders close to an interval order. *Discrete Applied Mathematics.* 6 (1983) 311-314.

Roubens, M. and Vincke, Ph. *Preference Modelling.* Springer-Verlag. (1985).

Roy, B. and Bouyssou, D. *Aide Multicritère à la Décision : Méthodes et cas.* Economica. Paris. (1993).

Roy, B. *Méthodologie Multicritère d'Aide à la Décision.* Economica. Paris. (1985).

Roy, B. Préférence, Indifférence, Incomparabilité. *Document du LAMSADE.* n°9. Université Paris Dauphine. (1980).

Vincke, Ph. *Multicriteria decision-aid.* Chichester, John Wiley & Sons (1992).

Wang, C. and Williams, A.C. The threshold weight of a graph. *Journal of Graph Theory,* Vol. 15, (1991) 235-249.

Yannakakis, M. The complexity of the partial order dimension problem, *SIAM J. Algebraic Discrete Methods* 3 (1982)351-358.

Zemirline, A. *Relations entre la théorie de la programmation linéaire et l'optimisation combinatoire.* Thèse de Doctorat d'Etat. USTHB, Alger. (1987).