RELIABILITY OF A TWO ELEMENT REDUNDANT SYSTEM WITH WAITING TIME DISTRIBUTION

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ABSTRACT. — In this paper the effect of waiting time on the reliability behaviour of two element parallel redundant system with exponential failure and general repair time distribution for both the elements, has been evaluated. The Laplace transforms of the various state probabilities have been obtained and the Main Time to System Failure (MTSF) has also been derived.

Introduction.

In the past, most of the reliability models on parallel redundancy have been formulated and solved assuming that the repair is instantaneous. This assumption does not seem to be realistic in practical situations, even if it is assumed that the fault detection and element switching is instantaneous and error free. However, it would be admitted beyond doubt that there always exists a time lag, between the failure of the element and its removal for repair. It also exists due to non-availability of the spares or the preoccupation of the repair facility. Therefore, to account for this time lag, the concept of waiting time distribution has been introduced in this paper.

To develop the mathematical model, consider a complex system comprised of two identical operating components A and B arranged in parallel redundancy, which only means that if one component fails, the other component continues performing the required mission, for which the system is intended. On failure, the component waits for repair and if this component is repaired before the failure of the second component, it is put back into the system.

The system is considered to have failed, when both the component fail. It has been assumed that the failure and waiting time distributions are governed by exponential law with constant rates λ and ν respectively, and repair follows general distribution with probability density $f(x)$. Further, the Mean Time to System Failure of such a system has also been evaluated.

Differential Equations Governing the Behaviour of the Complex System.

Define,

$P_0(t) \equiv$ The probability that, at time t, both the elements are working.

$P_W(t) \equiv$ The probability that, at time t, one element is working, and the other one while has failed, is waiting for repair.

$P_R(x, t)\,\Delta \equiv$ The probability that, at time t, one element is working and the other one which has failed, is being repaired and the elapsed repair time lies in the interval $(x, x + \Delta)$.

$P_F(t) \equiv$ The probability that, at time t, both the components are in the failed state and the system is considered to have failed.

$\eta(x)\,\Delta \equiv$ The first order probability that the element is repaired (i.e. made available for service) in the time interval $(x, x + \Delta)$ conditioned that it has not been repaired up to time x.

The relationship between $\eta(x)$ and the corresponding probability density function $f(x)$ is given by:

$$f(x) = \eta(x) \exp\left[-\int_0^x \eta(x)\, dx\right] \tag{1}$$

From continuity arguments, the following differential equations for the process may seem to hold good:

$$\left[\frac{\partial}{\partial t} + 2\lambda\right] P_0(t) = \int_0^\infty P_R(x, t)\eta(x)\, dx \tag{2}$$

$$\left[\frac{\partial}{\partial t} + \lambda + \nu\right] P_W(t) = 2\lambda P_0(t) \tag{3}$$

$$\left[\frac{\partial}{\partial t} + \frac{\partial}{\partial x} + \lambda + \eta(x)\right] P_R(x, t) = 0 \tag{4}$$

$$\frac{\partial}{\partial t} P_F(t) = \lambda \int_0^\infty P_R(x, t)\, dx + \lambda P_W(t) \tag{5}$$

The above equations are to be solved under the boundary conditions:

$$P_R(0, t) = \nu P_W(t) \tag{6}$$

It is assumed that initially both the components are working, i.e., $P_0(0) = 1$ and other state probability are zero.

Let the Laplace transform of the function $f(t)$ be represnted by $L\{f(t)\}$ or $\bar{f}(s)$ and is given by the expression:

$$\bar{f}(s) = L\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt$$

$$\text{Re}(s) > 0$$

Applying Laplace transforms and making use of the initial conditions, mentioned above equations (2) through (6), give:

$$[s + 2\lambda]\, \bar{P}_0(s) = 1 + \int_0^\infty \bar{P}_R(x, s)\, \eta(x)\, dx \qquad (7)$$

$$[s + \lambda + \nu]\, \bar{P}_W(s) = 2\lambda\, \bar{P}_0(s) \qquad (8)$$

$$[\frac{\partial}{\partial x} + s + \lambda + \eta(x)]\, \bar{P}_R(x, s) = 0 \qquad (9)$$

$$s\, \bar{P}_F(s) = \lambda \int_0^\infty \bar{P}_R(x, s)\, dx + \lambda\, \bar{P}_W(s) \qquad (10)$$

$$\bar{P}_R(0, s) = \nu\, \bar{P}_W(s)$$

Equation (9) on integration and simplification with the help of (11) gives:

$$\bar{P}_R(x, s) = \nu\, \bar{P}_W(s) \exp\, [-(s + \lambda)\, x - \int_0^x \eta(x)\, dx] \qquad (12,$$

On solving relations (7) through (10) and (12):

$$\bar{P}_0(s) = (s + \lambda + \nu) / k$$

$$\bar{P}_W(s) = 2\lambda / k \qquad (14)$$

$$\bar{P}_F(s) = 2\lambda^2\, [s + \lambda + \nu - \nu \bar{f}(s + \lambda)] / [ks(s + \lambda)] \qquad (15)$$

and

$$\bar{P}_R(s) = \int_0^\infty \bar{P}_R(x, s)\, dx = 2\lambda\mu\, [1 - \bar{f}(s + \lambda)] / [k(s + \lambda)] \qquad (16)$$

where

$$k = (s + 2\lambda)(s + \lambda + \mu) - 2\lambda\mu\, \bar{f}(s + \lambda)$$

and

$$\bar{f}(s + \lambda) = L\{e^{-\lambda x} f(x)\}$$

It may be verified that:

$$\bar{P}_0(s) + \bar{P}_W(s) + \bar{P}_R(s) + \bar{P}_F(s) = \frac{1}{s}$$

Relations (13) through (16) which give the various state probabilities may easily be inverted for specific values of the repair probability density.

Mean Time to System Failure.

Following Alven [3], the Mean Time to System Failure is given by:

$$\text{MTSF} = \int_0^\infty P_A(t)\, dt$$

where $P_A(t)$ is the probability that the system is working at time t, and the following condition holds:

$$\lim_{t\to\infty} t\, P_A(t) = 0$$

Again $P_A(t) = P_0(t) + P_W(t) + P_R(t) = 1 - P_F(t)$

$$\text{MTSF} = \int_0^\infty [1 - \bar{P}_F(t)]\, dt$$

$$= \lim_{T\to\infty} \int_0^T [1 - P_F(t)]\, dt$$

Following Lloyd and Lipow:

$$\text{MTSF} = \lim F(T) - F(0) = \lim_{T\to\infty} F(T)$$

$$\text{where} \quad F(T) = \int_0^T f(x)\, dx$$

From Laplace transform theory:

$$L\left[\int_0^T \{1 - P_F(t)\}\, dt\right] = \frac{1}{s} L\,[1 - P_F(t)]$$

$$= \frac{1}{s}\left[\frac{1}{s} - \bar{P}_F(s)\right] = \bar{F}(s)$$

Using Abel's corollary viz.:

$$\lim_{T\to\infty} F(T) = \lim_{s\to 0} s\,\bar{F}(s)$$

(If the limit on the right exists).

Therefore:

$$\lim_{T\to\infty} F(T) = \lim_{s\to 0} \left[\frac{1}{s} - \bar{P}_F(s)\right]$$

Substituting value of $\bar{P}_F(s)$ from (15), we get:

$$\text{MTSF} = \lim_{T\to\infty} F(T) = \frac{3(\lambda + \nu) - 2\nu\,\bar{f}(\lambda)}{2\lambda\,[\lambda + \nu - \nu\,\bar{f}(\lambda)]}$$

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REFERENCES

[1] Flehinger B.J. « Reliability through redundancy at various system level » I.B.M., Journal Research & Development, 2, 148 (1958)

[2] Lloyd, D.K. & Lipow M. « Reliability Management Methods and Mathematics » pp. 135-137 Prentice Hall, New Jersey 1962

[3] Von Alven, W.H. « Reliability Engineering » pp. 235-236, Prentice Hall, New Jersey 1964.