# GOAL PROGRAMMING APPROACHES FOR PRIORITIES IN SAATY'S ANALYTIC HIERARCHY PROCESS (AHP)

*Dieter K. Tscheulin;* Albert-Ludwigs-Universität Freiburg,
Wirtschaftswissenschaftliche Fakultät, Betriebswirtschaftliches Seminar IV,
Platz der Alten Synagoge 1, D-79085 Freiburg im Breisgau
Tel.: ++49-761-203-2409
Fax: ++49-761-203-2410
E-mail: tscheuli@vwl.uni-freiburg.de

*Jean-Marie Jacques,* Facultés Universitaires Notre-Dame de la Paix Namur,
Departement de Gestion de l'Entreprise, Rempart de la Vierge 8, B-5000 Namur
Tel.: ++32-81-724873
Fax: ++32-81-724840

## ABSTRACT

A variety of alternative estimation procedures have been proposed, from the use of different means (Saaty 1980) to the least square method (de Jong 1984, Jensen 1984) and linear programming (Korhonen/Wallenius 1990). The most common approach is the originally proposed eigenvector method (Saaty 1977), which is dominant in practical use, as numerous publications have demonstrated. Saaty (1990) compared the eigenvector method with the logarithmic least squares method and concluded with ten reasons for not using substitutes for the eigenvector, with „uniqueness" of solution as first of them. The necessity for further research on estimation procedures is pointed out by Zahedi (1986, p. 103) in his survey. Our paper proposes a goal programming approach different in its constructions from Korhonen and Wallenius' model as an alternative estimation procedure. By means of simulation results the approach's ability to produce unique solutions is demonstrated. As opposed to the other proposed methods, the goal programming approach offers the advantage of allowing a simultaneous estimation of all the decision elements under certain circumstances.

The authors wish to thank an unknown referee for his constructive comments on their paper.

## I Introduction

Analytic Hierarchy Process (AHP) (Saaty 1977, Saaty 1980) represents a method for deriving priority weights, which since is introduction by Saaty in 1977 has been subject to a variety of publications and applications. The application areas of AHP span from decision theory (i. e. Lootsma 1980, Whipple/Simmons 1987) to the problem of new product design (Tscheulin 1991). A Survey of the different applications has been presented by Zahedi (1986). Common to all applications is the hierarchical structure of the problem. The decision problem to be solved is defined as a hierarchy of single decision elements, between which certain relationships exist. Hence, relative priority weights are to be estimated for the single decision elements on each hierarchy level of the problem. A variety of alternative estimation procedures have been proposed spanning from the use of different means (Saaty 1980) to the least square method (de Jong 1984, Jensen 1984) and linear programming (Korhonen/Wallenius 1990). The most common approach is the originally proposed eigenvector method (Saaty 1977), which is dominant in practical use, as numerous publications have demonstrated. The necessity for further research on estimation procedures is pointed out by Zahedi (1986, p. 103) in his survey. This paper proposes a goal programming approach different in its construction from Korhonen and Wallenius' model as an alternative estimation procedure. By means of simulation results it is demonstrated, that our approach allows to produce unique solutions. As opposed to the other proposed methods, the goal programming approach offers the advantage of allowing a simultaneous estimation of all the decision elements under certain circumstances

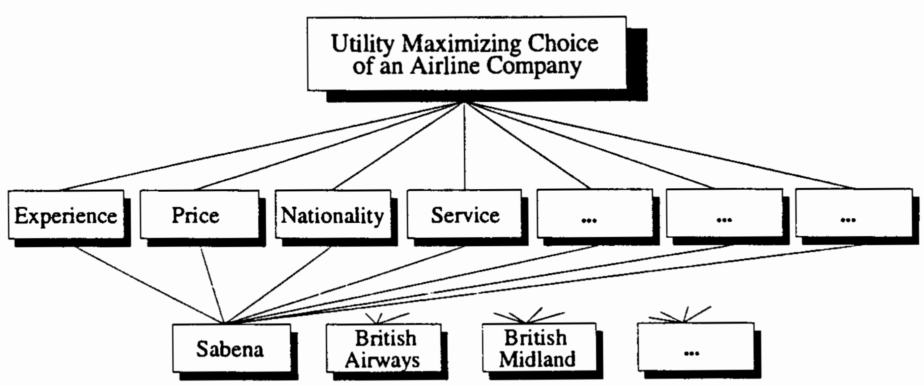
## II Structure of the Decision Problem

Analytic Hierarchy Process (AHP) structures the problem to be solved hierarchically. The number of hierarchy levels and elements in those hierarchy levels is variable. The elements of two neighbouring levels can be connected in two different ways (Saaty 1977, p. 258f., Saaty 1980, p. 42f.):

In the case of a complete hierarchy each element is connected to all elements in the level above. Figure 1 shows an example representing a decision problem consisting of three different hierarchy levels. The first hierarchy level has one single objective, the utility maximizing choice of an airline company. The elements in the second hierarchy level are given by the attributes that should be evaluated with respect to their significance for preference. In the example of fig. 1 these are the experience, service, nationality and price. The priority weights of these product attributes are determined by means of pairwise elicited ratio-scaled priority judgments. The respondent has to decide how much more important he/she considers the experience of an airline company compared to the price (or vice versa).

For measuring the strength of preference, Saaty (1977, p. 244ff., 1980, p. 53ff.) recommends the use of a 9-point-scale, because it can be easily comprehended by the respondent and has proved higher test/retest reliability scores than other scales. The third and bottom hierarchy level finally represents the alternatives which the respondent has to evaluate. The respondent is asked how much he/she prefers one alternative to another for each of the underlying product attributes.

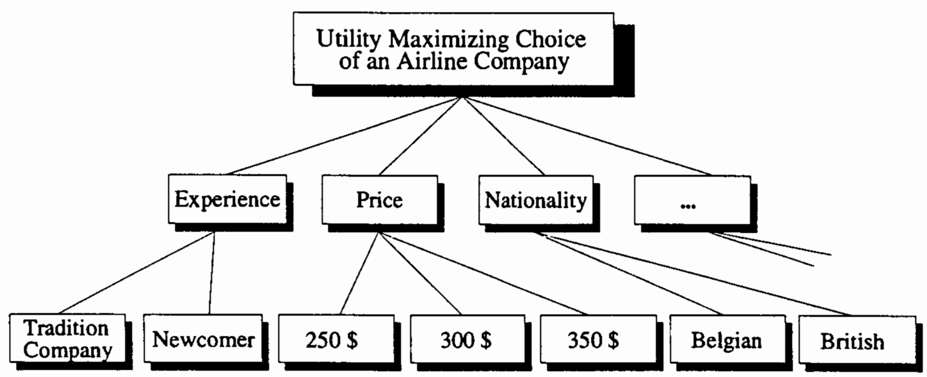
Fig. 1: A complete hierarchy for the choice of an airline company for a regular flight between Brussels and London




The objective of an AHP-study using complete hierarches is to determine ratio-scaled overall utility measurements for the alternative airline companies by means of the intermediate second hierarchy level representing the relevant product attributes.

In case of an incomplete hierarchy each element is connected to some, but not all, elements in the hierarchy level below.

Fig. 2: An incomplete hierarchy for the determination of priority weights of product attributes and attribute levels of a regular flight between Brussels and London




Again, fig. 2 shows a hierarchy consisting of three levels, where the first two levels are identical to those of fig. 1 for complete hierarchies. Instead of being connected to different product alternatives, the product attributes in the second hierarchy level are divided up into different attribute levels. Hence, the hierarchy illustrated in fig. 2 has the objective of determining priority weights for different attribute levels of a regular flight between Brussels and London. By means of a 9-point-scale the respondent has to evaluate, for example, by how much a belgian airline company is preferred to a british one (or vice versa). Hence, the goal of the analysis is to determine the respondent's part worth utilites of prespecified product attribute levels for the choice of a regular flight between Brussels and Londen, by means of the intermediate second hierarchy level representing the relevant product attributes.

## III Estimation of Priority Weights with Linear Programming

Let n be the number of decision elements $A_1, A_2, \ldots\ldots, A_n$, the pairwise weighting ratios of which are known from the matrix of pairwise comparison judgements.

$$
\begin{array}{c|cccc}
 & A_1 & A_2 & \ldots & A_n \\
\hline
A_1 & w_1/w_1 & w_1/w_2 & \ldots & w_1/w_n \\
A_2 & w_2/w_1 & w_2/w_2 & \ldots & w_2/w_n \\
\cdot & \cdot & \cdot & \ldots & \cdot \\
\cdot & \cdot & \cdot & \ldots & \cdot \\
\cdot & \cdot & \cdot & \ldots & \cdot \\
A_n & w_n/w_1 & w_n/w_2 & \ldots & w_n/w_n
\end{array}
= A \qquad (1)
$$

$$a_{ij} = w_i/w_j \qquad i, j = 1, \ldots, n \qquad (2)$$

The following equations (3) to (8) describe the eigenvector method as in the originally proposed estimation procedure:

Let $\Omega = \{(1, 2), (1, 3), \ldots, (1, n), (2, 3), \ldots, (2, n), \ldots, (n\text{-}1, n)\}$ be the set of paired comparison judgments, where for each pair (i, j) decision element $A_i$'s preference is judged greater than or equal to that of decision element $A_j$.

From (2) $\qquad a_{ij} = w_i/w_j,\ Q = \{1, \ldots, n\}$, we obtain

$$a_{ij} \cdot (w_j/w_i) = 1 \qquad (3)$$

and consequently

$$\sum_{j=1}^{n} a_{ij} \cdot (w_j/w_i) = n \qquad (4)$$

or

$$\sum_{j=1}^{n} a_{ij} \cdot w_j = n \cdot w_i \qquad (5)$$

In matrix theory equation (5) can be expressed as

$$\begin{vmatrix} w_1/w_1 & w_1/w_2 & \dots & w_1/w_n \\ w_2/w_1 & w_2/w_2 & \dots & w_2/w_n \\ . & . & \dots & . \\ . & . & \dots & . \\ . & . & \dots & . \\ w_n/w_1 & w_n/w_2 & \dots & w_n/w_n \end{vmatrix} \begin{vmatrix} w_1 \\ w_2 \\ . \\ . \\ . \\ w_n \end{vmatrix} = n \cdot \begin{vmatrix} w_1 \\ w_2 \\ . \\ . \\ . \\ w_n \end{vmatrix} \qquad (6)$$

or

$$A \cdot w = n \cdot w \qquad (7)$$

or

$$(A - n \cdot I) \cdot w = 0 \qquad (8)$$

and represents an eigenvector problem where w is an eigenvector of A with the corresponding eigenvalue n. Supposing that the respondent's priority judgments are no longer consistent, (8) can be written as

$$(A - \lambda_{max} \cdot I) \cdot w = 0 \qquad (9)$$

The eigenvector w corresponding to the largest eigenvalue $\lambda_{max}$ contains the priority weights of the decision elements in terms of the corresponding element in the hierarchy level above. As an index of consistency Saaty (1980, p. 21) derives C.I. = $(\lambda_{max} - n)/(n - 1)$.

The following section presents a goal-programming approach which persues the same objective as the eigenvector method, that of estimation relative priority weights of decision elements with regard to an element in the hierarchy level above.

From (2) we obtain

$$a_{ij} \cdot w_j = w_i \tag{10}$$

or

$$a_{ij} \cdot w_j - w_i = 0 \tag{11}$$

If the respondent's priority judgments are consistent, a vector of relative priority weights fulfilling the constraints (11) can be estimated even if we exclude the trivial solution, where all $w_i$, $w_j = 0$. This vector of relative priority weights would be identical to that obtained by the eigenvector method when priority judgments are consistent.

If we turn to the more practical case, where the respondent's priority judgments are not fully consistent, it is no longer possible to fulfill all all conditions of type (11). For this reason equation (11) will be supplemented by slack- and surplus-variables:

$$a_{ij} \cdot w_j - w_i - z_{ij}^{+} + z_{ij}^{-} = 0 \quad \text{for } (i, j) \in \Omega \tag{12}$$

$z_{ij}^{+}$ and $z_{ij}^{-}$ guarantee that equation (12) can always be fulfilled, even in case of inconsistency. Hence, the objective criterion is to minimize the sum of deviations from all constraints of type (12).

To exclude the trivial solution, all $w_i$, $w_j = 0$, an additional constraint is necessary to fix the sum of all relative priority weights equal to one. Consequently, the eigenvector method's affordable normalization of the vector of relative priority weights is superfluous.

In order to avoid a dually degenerate solution (as shown by simulation results in table 1), constraint (15) forces the sum of slack and surplus variables to be equal for each constraint of type (14).

The resulting LP has the following shape:

$$\text{Minimize } B = \sum_{(i,j\in\Omega)} z_{ij}^{+} + \sum_{(i,j\in\Omega)} z_{ij}^{-} \tag{13}$$

subject to

$$a_{ij} \cdot w_j - w_i - z_{ij}^{+} + z_{ij}^{-} = 0 \quad \text{for } (i, j) \in \Omega \text{ and } i, j \in Q \tag{14}$$

$$(-z_{ij_m}^{+} + z_{ij_m}^{-}) = (-z_{ij_n}^{+} + z_{ij_n}^{-}) \quad \text{for } (i, j_m), (i, j_n) \in \Omega \text{ and } j_m \neq j_n \tag{15}$$

$$\sum_i w_i = 1 \quad \text{for } i \in Q \tag{16}$$

$$w_i \geq 0 \quad \text{for } i \in Q \tag{17}$$

$$z_{ij}^{+}, z_{ij}^{-} \geq 0 \quad \text{for } i, j \in \Omega \tag{18}$$

$w_i, w_j$ = Priority weight of decision element i and j respectively related to the element in the hierarchy level above

$a_{ij}$ = the respondent's priority judgment in terms of the differences in preference between decision element i and decision element j related to the element in the hierarchy level above

The goal progamming approach described by (13) to (18) guarantees a global optimal solution. The resulting objective value B is a measure of the consistency of the respondent's judgments.

Table 1 presents simulation results for different levels of consistency. While brand A is assumed to be three times more preferred than brand B, and brand B three times more than band C, preference between brand A and brand C is simulated from plus nine (perfect consistency) to minus nine. As table 1 shows, all approaches, the eigenvector method as well as the presented goal programming approach and the same goal programming approach neglecting constraint (15), produces identical preference weights for the case of consistency. For the case of inconsistency it can be seen that the goal programming approach produces higher preference weights for the dominating brand as long as the level of inconsistency does not violate the order of transitivity, i. e. brand A is still more preferred than brand C. The observed gap between the preference weights for the dominating brand increases with increasing inconsistency. For case six where transitivity is reversed, i. e. brand C is three times more preferred than brand A, both methods again produce identical results.

As table 1 also shows, the goal programming approach's constraint (15) decreases the model's index of fit, but is necessary to avoid dual degeneration and results not matching the condition of face validity.

Table 1: Simulation results of estimated preference weights for different priority judgments with corresponding different levels of inconsistency
(Goal neg. = Goal Programming Approach neglecting constraint (15))

| Matrix of pairwise comparison judgments | | | Estimation procedure | estimated preference weights | | | consistency index (see pp. 5 and 6) |
|---|---|---|---|---|---|---|---|
| 1 | 3 | 9 | Eigenvector | 0,692 | 0,231 | 0,077 | 0,000 |
| 1/3 | 1 | 3 | Goal progr. | 0,692 | 0,231 | 0,077 | 0,000 |
| 1/9 | 1/3 | 1 | Goal neg. | 0,692 | 0,231 | 0,077 | 0,000 |
| 1 | 3 | 7 | Eigenvector | 0,669 | 0,243 | 0,088 | 0,006 |
| 1/3 | 1 | 3 | Goal progr. | 0,672 | 0,236 | 0,090 | 0,036 |
| 1/7 | 1/3 | 1 | Goal neg. | 0,677 | 0,225 | 0,096 | 0,006 |
| 1 | 3 | 5 | Eigenvector | 0,637 | 0,258 | 0,105 | 0,033 |
| 1/3 | 1 | 3 | Goal progr. | 0,644 | 0,244 | 0,111 | 0,266 |
| 1/5 | 1/3 | 1 | Goal neg. | 0,652 | 0,217 | 0,130 | 0,173 |
| 1 | 3 | 3 | Eigenvector | 0,584 | 0,281 | 0,135 | 0,117 |
| 1/3 | 1 | 3 | Goal progr. | 0,600 | 0,257 | 0,142 | 0,514 |
| 1/3 | 1/3 | 1 | Goal neg. | 0,600 | 0,200 | 0,200 | 0,400 |
| 1 | 3 | 1 | Eigenvector | 0,460 | 0,319 | 0,221 | 0,483 |
| 1/3 | 1 | 3 | Goal progr. | 0,520 | 0,280 | 0,200 | 0,960 |
| 1 | 1/3 | 1 | Goal neg. | 0,692 | 0,231 | 0,077 | 0,615 |
| 1 | 3 | 1/3 | Eigenvector | 0,333 | 0,333 | 0,333 | 1,149 |
| 1/3 | 1 | 3 | Goal progr. | 0,333 | 0,333 | 0,333 | 2,000 |
| 3 | 1/3 | 1 | Goal neg. | Dual Degeneration | | | 2,000 |
| 1 | 3 | 1/5 | Eigenvector | 0,278 | 0,330 | 0,391 | 1,585 |
| 1/3 | 1 | 3 | Goal progr. | 0,245 | 0,358 | 0,396 | 2,490 |
| 5 | 1/3 | 1 | Goal neg. | 0,047 | 0,714 | 0,238 | 2,095 |
| 1 | 3 | 1/7 | Eigenvector | 0,245 | 0,325 | 0,431 | 1,923 |
| 1/3 | 1 | 3 | Goal progr. | 0,194 | 0,373 | 0,432 | 2,776 |
| 7 | 1/3 | 1 | Goal neg. | 0,034 | 0,724 | 0,241 | 2,137 |
| 1 | 3 | 1/9 | Eigenvector | 0,221 | 0,319 | 0,460 | 2,205 |
| 1/3 | 1 | 3 | Goal progr. | 0,160 | 0,382 | 0,456 | 2,962 |
| 9 | 1/3 | 1 | Goal neg. | 0,027 | 0,729 | 0,243 | 2,162 |

The number of relative priority weights resulting from the different goal programming approaches of type (13) to (18) can be processed analogically to the eigenvector method in order to achieve a single vector of priority weights for the decision elements in the hierarchy's bottom level with respect to the hierarchy's overall objective.

In the following section, a goal-programming approach is presented which enables a simultaneous estimation over all hierarchy levels for the case of an incomplete hierarchy as it is relevant for the problem of optimal product design. In this case the elements in the second hierarchy level symbolize the product attributes, the elements in the third hierarchy level the corresponding attribute levels. Constraint (14) will be supplemented by a constraint forcing the sum of priority weights of one product attribute's attribute levels to be in a relation to the sum of priority weights of another product attribute's attribute levels according to the respondent's judgment of preference. Hence, the resulting goal programming approach has the following shape:

$$\text{Minimize } B = \sum_{(i,j \in \Omega_k)} \sum_{k} z_{ijk}^{+} + \sum_{(i,j \in \Omega_k)} \sum_{k} z_{ijk}^{-} + \sum_{(k,l \in \Omega_0)} z_{kl}^{+} + \sum_{(k,l \in \Omega_0)} z_{kl}^{-} \qquad (19)$$

subject to

$$a_{ijk} \cdot w_{jk} - w_{ik} - z_{ijk}^{+} + z_{ijk}^{-} = 0 \quad \text{for } i < j,\ (i,j) \in \Omega_k,\ k \in \Omega_0 \qquad (20)$$

$$(-z_{ij_m k}^{+} + z_{ij_m k}^{-}) = (-z_{ij_n k}^{+} + z_{ij_n k}^{-}) \text{ for } (i, j_m), (i, j_n) \in \Omega_k \text{ with } j_m \neq j_n \text{ and } k \in \Omega_0 \qquad (21)$$

$$a_{kl} \cdot \sum w_{ik} - \sum w_{il} - z_{kl}^{+} + z_{kl}^{-} = 0 \quad \text{for } k < l,\ (k,l \in \Omega_0) \qquad (22)$$

$$(-z_{kl_o}^{+} + z_{kl_o}^{-}) = (-z_{kl_p}^{+} + z_{kl_p}^{-}) \text{ for } (k, l_o), (k, l_p) \in \Omega_0 \text{ with } l_o \neq l_p \qquad (23)$$

$$\sum_{i} w_{ik} = 1 \qquad \text{for } (i, j) \in \Omega_k,\ k \in \Omega_0 \qquad (24)$$

$$w_{ik},\ w_{il} \geq 0 \qquad (25)$$

$$z_{ijk}^{+}, z_{ijk}^{-}, z_{kl}^{+}, z_{kl}^{-} \geq 0 \text{ for } (i,j) \in \Omega_k,\ k, l \in \Omega_0 \qquad (26)$$

$i, j$ = Indices for the decision elements in the 3rd hierarchy level

$k, l$ = Indices for the decision elements in the 2nd hierarchy level

$w_{ik}$ = Priority weight of decision element i of the 3rd hierarchy level referring to decision element k of the 2nd hierarchy level with regard to the hierarchy's overall objectice

$a_{ijk}$ = Priority judgment referring to the differences of preference between decision element i and decision element j of the 3rd hierarchy level with regard to he corresponding decision element k of the 2nd hierarchy level

$a_{kl}$ = Priority judgment referring to the difference of preference between decision element k and decision element 1 of the 2nd hierarchy level with regard to the hierarchy's overall objective

$\Omega_k$ = Set of paired comparison judgments referring to the differences of preference between decision element i and decision element j of the 3rd hierarchy level with regard to the corresponding decision element k of the 2nd hierarchy level

$\Omega_0$ = Set of paired comparison judgments referring to the differences of preference between decision element k and decision element 1 of the 2nd hierarchy level with regard to the hierarchy's overall objective

The goal programming approach described by equations (19) to (26) enables a simultaneous estimation of the interesting priority weights of the elements in the hierarchy's bottom level, referring to the hierarchy's overall objective. For complete hierarchies too, a simultaneous estimation of all hierarchy elements is conceivable. Certainly, this would enforce the use of nonlinear programming because of the multiplicative relationships between the different hierarchy levels.

As another alternative for the originally proposed eigenvector method, a goal programming approach was presented as an estimation procedure for Saaty's Analytic Hierarchy Process (AHP), enabling derivation of global optimal relative priority weights. For the exceptional cases of consistency as well as perfect intransitivity (i. e. A three times better than B, B three times better than C, C three times better than A) the results are identical to those of the eigenvector method. By means of simulation results it was shown that in all other cases both approaches produce slightly different preference weights.

Future research should investigate which of the two approaches has greater predictive validity in case of inconsistency. For incomplete hierarchies it was shown how linear programming enables a simultaneous estimation of the priority weights of all decision elements. For this case too, an empirical investigation of the advantages and disadvantages will be necessary.

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# ALBERT-LUDWIGS-UNIVERSITÄT FREIBURG

PROF. DR. DIETER K. TSCHEULIN

ABTEILUNG FÜR BETRIEBSWIRTSCHAFTSLEHRE IV
BETRIEBSWIRTSCHAFTLICHES SEMINAR
EUROPAPLATZ 1
D-79085 FREIBURG IM BREISGAU

Tel. +49-761/203-2409
Fax. +49-761/203-2410
E-Mail: tscheulin@vwl.uni-freiburg.de

Prof. Dr. Dieter K. Tscheulin, Universität Freiburg, Europaplatz 1, 79085 Freiburg

JORBEL
Belgian Journal of Operations Research,
Statistics and Computer Science
Renaissance Ave 30

B-1040 Brussel

Freiburg, 25.07.97

Ref: JORBEL / 158 / 93

Dear editors,

please find enclosed the revised version of our paper „Goal Programming Approaches for Priorities in SAATY's Analytic Hierarchy Process (AHP)". We apologize for the delay in our submission of this errorfree version. We hope you are still interested in publishing it.

Thank you for your understanding, sincerely yours

(Prof. Dr. Dieter K. Tscheulin)