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Fuzzy Multi-Objective Linear Programming

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Abstract

The paper presents an overview of methods for solving three classes of fuzzy multi-objective linear programming problems: flexible programming, multi-objective linear programming with fuzzy coefficients and flexible multi-objective linear programming with fuzzy coefficients. They are using fuzzy sets to deal with uncertainty connected either with preferential information concerning satisfaction of goals and constraints, or with an incomplete or vague state of information used to build the linear programming model, or with both these kinds of uncertainty together, respectively. Special attention is paid to interpretation of fuzzy goals and fuzzy constraints, to interpretation of fuzzy relations and to the character of interaction with the decision maker.

Key words: Multi-Objective Linear Programming under Uncertainty, Uncertainty Modeling, Fuzzy Sets, Multi-Criteria Decision Analysis, Interactive Procedures.

1. Introduction

Linear programming (LP) is today one of the most frequently applied OR techniques in real-world problems. This is due to the powerful simplex method able to handle thousands of variables and constraints. Because of this popularity, many efforts have been devoted to generalizations introducing to the LP models some new realistic aspects of decision problems while preserving the applicability of the simplex method. Probably the most striking example of this generalization is the Fuzzy Multi-Objective Linear Programming (FMOLP) which extends the LP model in two important aspects:

multiple objective functions representing multiple points of view used for evaluation of feasible solutions,

uncertainty inherent to information used in the modeling and solving stage.

A general model of the FMOLP problem can be presented as the following system:

$$[\tilde{c}_1\boldsymbol{x}, \tilde{c}_2\boldsymbol{x}, \ldots, \tilde{c}_k\boldsymbol{x}] \rightarrow m\tilde{i}n \quad (1)$$

$$\text{subject to } \ \tilde{\boldsymbol{a}}_i\boldsymbol{x} \tilde{\leq} \tilde{b}_i \quad i=1,\ldots,m \quad (2)$$

$$\boldsymbol{x} \geq 0 \quad (3)$$

where $\tilde{c}_l = [\tilde{c}_{l1}, \ldots, \tilde{c}_{ln}]$ (l=1,...,k), $\boldsymbol{x} = [x_1, \ldots, x_n]^T$, $\tilde{\boldsymbol{a}}_i = [\tilde{a}_{i1}, \ldots, \tilde{a}_{in}]$ (i=1,...,m). The coefficients with the sign of wave are, in general, fuzzy numbers, i.e. convex continuous fuzzy subsets of the real line. The wave over ***min*** and relation ≤ "fuzzify" their meaning. Condition (2) and (3) define a set of feasible solutions (decisions) X. An additional information completing (1) is a set of fuzzy aspiration levels on particular objectives, thought of as goals, denoted by $\tilde{g}_1, \ldots, \tilde{g}_k$.

There are three important special cases of the above problem that gave birth to the following classes of problems:

flexible programming,

multi-objective linear programming (MOLP) with fuzzy coefficients,

flexible MOLP with fuzzy coefficients.

In flexible programming, coefficients are crisp but there is a fuzzified relation $\tilde{\leq}$ between objective functions and the goals, and between left- and right-hand sides of the constraints. This means that the goals and constraints are fuzzy ("soft") and the key question is the degree of satisfaction. In MOLP with fuzzy coefficients all the coefficients are, in general, fuzzy numbers and the key question is a representation of relation ≤ between fuzzy left- and right-hand sides of the constraints. Flexible MOLP with fuzzy coefficients concerns the most general form of (1)-(3) and combines the two key questions of the previous problems.

The two first classes of FMOLP problems use different semantics of fuzzy sets while the third class combines the two semantics. In flexible programming, fuzzy sets are used to express preferences concerning satisfaction of flexible constraints and/or attainment of goals. This semantics is especially important for exploiting information in decision making. The gradedness introduced by fuzzy sets refines the simple binary distinction made by ordinary constraints. It also refines the crisp specification of goals and „all-or-nothing" decisions. Constraint satisfaction algorithms, optimization techniques and multi-criteria decision analysis are typically involving flexible requirements which can be represented by fuzzy relations.

In MOLP with fuzzy coefficients, the semantics of fuzzy sets is related to the representation of incomplete or vague states of information under the form of possibility distributions. This view of fuzzy sets enables representation of imprecise or uncertain information in mathematical models of decision problems considered in operations research. In models formulated in terms of mathematical programming, the imprecision and uncertainty of information (data) is taken into account through the use of fuzzy numbers or fuzzy intervals instead of crisp coefficients. It involves fuzzy arithmetic and other mathematical operations on fuzzy numbers that are defined with respect to the famous Zadeh's extension principle.

In flexible MOLP with fuzzy coefficients, the uncertainty and the preference semantics are encountered together. This is typical for decision analysis and operations research where, in order to deal with both uncertain data and flexible requirements, one can use a fuzzy set representation.

The aim of this paper is not to make another survey (see, e.g., [14, 16, 18, 25, 28, 30, 34]) - it would need much more place than available. Instead, we wish to make a tutorial characterization of the three classes of problems and solution methods. This is done in the three subsequent sections followed by a final section with concluding remarks.

2. Flexible Programming

Flexible programming has been considered for the first time by Tanaka et al. [38] with respect to single-objective linear programming. It is based on a general principle of Bellman and Zadeh [1] defining the concept of *fuzzy decision* as an intersection of *fuzzy goals* and *fuzzy constraints*. A fuzzy goal corresponding to objective $\boldsymbol{c}_l \boldsymbol{x}$ is defined as a fuzzy set in X; its membership function $\mu_l : X \to [0,1]$ characterizes the decision maker's aspiration of making $\boldsymbol{c}_l \boldsymbol{x}$ "essentially smaller or equal to g_l". A fuzzy constraint corresponding to $\boldsymbol{a}_i \boldsymbol{x} \lesssim b_i$ is also defined as a fuzzy set in X; its membership function $\mu_i : X \to [0,1]$ characterizes the degree of satisfaction of the i-th constraint.

In order to define the membership function $\mu_i(\boldsymbol{x})$ for the i-th fuzzy constraint, one has to know the tolerance margin $d_i \geq 0$ for the right-hand side b_i (i=1,...,m):

$$\mu_i(\boldsymbol{x}) = \begin{cases} 1 & \text{for } \boldsymbol{a}_i \boldsymbol{x} \leq b_i \\ \text{strictly decreasing from 1 to 0} & \text{for } b_i < \boldsymbol{a}_i \boldsymbol{x} < b_i + d_i \\ 0 & \text{for } \boldsymbol{a}_i \boldsymbol{x} \geq b_i + d_i \end{cases} \qquad (4)$$

Specifying a membership level α, $\alpha \in [0,1]$, Tanaka et al. [38] have restricted the set of feasible solutions of each fuzzy constraint to the crisp set

$$X_\alpha^i = \{x \mid \mu_i(x) \geq \alpha\}, \qquad i=1,\ldots,m.$$

Then, the set of feasible solutions of a flexible programming problem is $X_\alpha = \bigcap_{i=1}^{m} X_\alpha^i$.

The single objective function is replaced by the fuzzy goal $\mu_G(x) = \dfrac{\min_{x \in X_0} \{cx\}}{cx}$.

To get an optimal solution one has to determine the optimal pair (α^*, x^*) such that

$$\min(\alpha^*, \mu(x^*)) = \sup_\alpha \min\left(\alpha, \max_{x \in X_\alpha} \{\mu_G(x)\}\right) \tag{5}$$

If the optimal α^* was determined a priori, the problem (5) could be reduced to a crisp mathematical programming problem where the objective was to find x^* that maximizes $\mu_G(x)$ on the set X_{α^*}. In the general case, an iterative algorithm is necessary where, beginning with any $\alpha_1 \in [0,1]$, the values α_k and $\max_{x \in X_{\alpha_k}} \{\mu_G(x)\}$ converge to the optimum step by step.

Zimmermann [43] has proposed a more integrative approach to flexible programming allowing consideration of multiple goals and constraints on a common ground. An aspiration level g_l and a tolerance margin $d_l \geq 0$ have to be assumed for the l-th goal (l=1,...,k) when assessing the membership function $\mu_l(x)$ as:

$$\mu_l(x) = \begin{cases} 1 & \text{for } c_l x \leq g_l \\ \text{strictly decreasing from 1 to 0} & \text{for } g_l < c_l x < g_l + d_l \\ 0 & \text{for } c_l x \geq g_l + d_l \end{cases} \tag{6}$$

According to the principle of Bellman and Zadeh, the set of fuzzy decisions is characterized by an aggregation of the component membership functions. If a conjunctive minimum operator were used for the aggregation, the membership function would be:

$$\mu_D(x) = \min_{l,i} \{\mu_l(x), \mu_i(x)\} \tag{7}$$

Then, the problem of finding the best decision (solution) boils down to the following optimization problem:

$$\mu_D(x) \to \max \tag{8}$$
$$\text{subject to } \; x \geq 0$$

The value of the aggregated function $\mu_D(x)$ can be interpreted as the overall degree of satisfaction of the decision maker with k fuzzy goals and m fuzzy constraints.

In case of minimum operator (7), problem (8) becomes:

$$\begin{aligned} &\nu \to max \\ &\text{subject to } \ \nu \le \mu_l(x) \ \ l=1,\ldots,k \\ &\qquad\qquad\quad \nu \le \mu_i(x) \ \ i=1,\ldots,m \\ &\qquad\qquad\quad x \ge 0 \end{aligned} \tag{9}$$

In [43, 44], Zimmermann has applied linear membership functions (4), (6) in problem (9) thus getting an ordinary LP problem. He also proposed to use the product operator instead of minimum, however, then (8) becomes nonlinear even if linear membership functions are used.

A comprehensive review of various propositions for modeling the functions $\mu_D(x)$ can be found in [36, 45].

Knowing the membership functions $\mu_l(x)$ (l=1,...,k) for fuzzy goals, one can define a Pareto optimal solution in the space of membership values, called *M-Pareto optimal solution* [30].

Definition (*M-Pareto optimal solution*) Solution x^* is said to be M-Pareto optimal if and only if there does not exist another $x \in X$ such that $\mu_l(x) \ge \mu_l(x^*)$, l=1,...,k, with strict inequality holding for at least one l.

The concept of M-Pareto optimal solutions was at the origin of several interactive methods proposed for flexible programming (see [28, 30]). In these methods, the decision maker determines membership functions for fuzzy goals and then specifies reference levels for the membership functions, denoted by $\overline{\mu}_l$ (l=1,...,k). Assuming some minimum levels for membership functions of fuzzy constraints, denoted by t_i (i=1,...,m), one gets the following optimization problem:

$$\begin{aligned} &\max_l \{\overline{\mu}_l - \mu_l(x)\} \to min \\ &\text{subject to } \ \mu_i(x) \ge t_i \ \ i=1,\ldots,m \\ &\qquad\qquad\quad x \ge 0 \end{aligned}$$

which is equivalent to:

$$\begin{aligned} &\nu \to min \\ &\text{subject to } \ \nu \ge \overline{\mu}_l - \mu_l(x) \qquad l=1,\ldots,k \\ &\qquad\qquad\quad \mu_i(x) \ge t_i \qquad i=1,\ldots,m \\ &\qquad\qquad\quad x \ge 0 \end{aligned} \tag{10}$$

Again, problem (10) becomes an ordinary LP problem when all membership functions are linear.

This approach is interactive in the sense that the reference levels can be changed from one iteration to another, as well as the membership functions of fuzzy goals.

3. MOLP with Fuzzy Coefficients

All fuzzy coefficients of the FMOLP problem are given in a convenient form of *L-R* fuzzy numbers [11]. An *L-R* (flat) fuzzy number $\tilde{a} = \left(a^L, a^R, \alpha^L, \alpha^R\right)_{LR}$ is defined by the membership function:

$$\mu_{\tilde{a}}(r) = \begin{cases} L((a^L - r)/\alpha^L) & \text{for } r \le a^L \\ 1 & \text{for } a^L \le r \le a^R \\ R((r - a^R)/\alpha^R) & \text{for } r \ge a^R \end{cases}$$

where L and R are symmetric bell-shaped reference functions which are strictly decreasing in [0,1] and such that $L(0) = R(0) = 1$, $L(1) = R(1) = 0$; $[a^L, a^R]$ is an interval of the most possible values, and α^L and α^R are nonnegative left and right "spreads" of $\tilde{a}$, respectively.

Experience indicates that an expert can describe the precise form of a fuzzy number only rarely. Therefore, as a practical way of getting suitable membership functions of fuzzy coefficients, Rommelfanger [24] has proposed that the expert begins with the specification of some prominent membership levels α and associates them with special meanings. After that the expert is expected to specify values which belong to the selected membership levels.

α=1: $\mu_{\tilde{a}}(r) = 1$ means that value r certainly belongs to the set of possible values,

α=λ: $\mu_{\tilde{a}}(r) \ge \lambda$ means that the expert estimates that value r with $\mu_{\tilde{a}}(r) \ge \lambda$ has a good chance of belonging to the set of possible values,

α=ε: $\mu_{\tilde{a}}(r) < \varepsilon$ means that value r with $\mu_{\tilde{a}}(r) < \varepsilon$ has only a very little chance of belonging to the set of possible values, i.e. the expert is willing to neglect the corresponding values of r with $\mu_{\tilde{a}}(r) < \varepsilon$.

For example, it is reasonable to assume that λ=0.6, ε=0.1.

For the sake of clarity, let us assume that the reference functions of all fuzzy coefficients are of two kinds only: L and R. It should be specified, moreover, that all arithmetic operations on fuzzy numbers taking place in (1), (2) are extended operations in the sense of Zadeh's extension principle [42]:

$$f_{\tilde{a}*\tilde{b}}(r) = \sup_{r=y*z} T(f_{\tilde{a}}(y), f_{\tilde{b}}(z)), \quad r \in \Re \tag{11}$$

where $*$ is a real operation $* : \Re \times \Re \to \Re$ and T: [0,1]×[0,1]→[0,1] is any given t-norm.

For any $x \geq 0$, the left-hand side of the i-th constraint and the value of the l-th objective function can be summarized to the following fuzzy numbers:

$$\tilde{a}_i x = \left(a_i^L x, a_i^R x, \alpha_i^L x, \alpha_i^R x\right)_{LR} \quad i = 1,\dots,m,$$
$$\tilde{c}_l x = \left(c_l^L x, c_l^R x, \gamma_l^L x, \gamma_l^R x\right)_{LR} \quad l = 1,\dots,k.$$

In the literature, the *min* t-norm is generally applied. Then,

$$a_i^L x = \sum_{j=1}^{n} a_{ij}^L x_j, \quad c_l^L x = \sum_{j=1}^{n} c_{lj}^L x_j \tag{12i}$$

$$a_i^R x = \sum_{j=1}^{n} a_{ij}^R x_j, \quad c_l^R x = \sum_{j=1}^{n} c_{lj}^R x_j \tag{12ii}$$

$$\alpha_i^L x = \sum_{j=1}^{n} \alpha_{ij}^L x_j, \quad \gamma_l^L x = \sum_{j=1}^{n} \gamma_{lj}^L x_j \tag{12iii}$$

$$\alpha_i^R x = \sum_{j=1}^{n} \alpha_{ij}^R x_j, \quad \gamma_l^R x = \sum_{j=1}^{n} \gamma_{lj}^R x_j \tag{12iv}$$

Obviously, the spreads of these fuzzy numbers extend when number and size of variables increase. The simple addition of the spreads of fuzzy coefficients corresponds to the assumption that their uncertainty comes from independent sources. This is not realistic in many practical situations. For getting a more realistic extended addition of the left-hand sides of fuzzy constraints and of fuzzy objectives, Rommelfanger and Keresztfalvi [27] recommend the use of Yager's parameterized t-norm:

$$T_p(t_1,\dots,t_s) = max\left\{0, 1 - \left(\sum_{i=1}^{s}(1-t_i)^p\right)^{1/p}\right\} \tag{13}$$
$$t_1,\dots,t_s \in [0,1], \quad p > 0$$

Then, $a_i^L x, a_i^R x, c_l^L x, c_l^R x$ are calculated according to (12i) and (12ii), however, the spreads $\alpha_i^L x, \alpha_i^R x, \gamma_l^L x, \gamma_l^R x$ are calculated according to a new, less cumulative formula:

$$\alpha_i^L x = \left(\sum_{j=1}^{n}\left(\alpha_{ij}^L x_j\right)^q\right)^{1/q}, \quad \alpha_i^R x = \left(\sum_{j=1}^{n}\left(\alpha_{ij}^R x_j\right)^q\right)^{1/q}$$

$$\gamma_i^L x = \left(\sum_{j=1}^{n}\left(\gamma_{ij}^L x_j\right)^q\right)^{1/q}, \quad \gamma_i^R x = \left(\sum_{j=1}^{n}\left(\gamma_{ij}^R x_j\right)^q\right)^{1/q}$$

where $q = \dfrac{p}{p-1} \geq 1.$

Coming back to the MOLP problem with fuzzy coefficients, we have to answer the question how to interpret the relation between fuzzy left- and right-hand side of the constraints. If constraints (2) were transformed to equality constraints (by addition of slack variables on the left) then the equality relation could be interpreted in terms of a weak inclusion of fuzzy sets [10, 19]:

$$\widetilde{a}_i x \subseteq \widetilde{b}_i \quad i = 1,\dots,m \tag{14}$$

It says that the region of possible values of the left-hand side should be contained in the tolerance region of the right-hand side. The LP problem with constraints (14) is called *robust programming problem.*

Each constraint (14) is then reduced to four deterministic constraints:

$$\begin{aligned} & a_i^L x \geq b_i^L, \quad a_i^R x \leq b_i^R \\ & a_i^L x - \alpha_i^L x \geq b_i^L - \beta_i^L \\ & a_i^R x + \alpha_i^R x \leq b_i^R + \beta_i^R, \quad \text{for } i = 1,\dots,m \end{aligned} \tag{15}$$

where $\widetilde{b}_i = \left(b_i^L, b_i^R, \beta_i^L, \beta_i^R\right)_{LL}$ or $\widetilde{b}_i = \left(b_i^L, b_i^R, \beta_i^L, \beta_i^R\right)_{RR}$, $i = 1,\dots, m.$

In order to transform fuzzy objectives into deterministic equivalents, one can consider a "middle" value of $\widetilde{c}_l x$ at some level $\xi \in [0,1]$, $l = 1,\dots, k$. The "middle" can be understood [7] as a combination of the most possible values $c_l^L x$ and $c_l^R x$, and of the smallest and the greatest (extreme) values at possibility level ξ. Thus, the objectives (1) become:

$$[z_1(x), z_2(x),\dots, z_k(x)] \rightarrow min \tag{16}$$

where $z_l(x) = w_1 c_l^L - w_2 \gamma_l^L x\, L^{-1}(\xi) + w_3 c_l^R x + w_4 \gamma_l^R x\, R^{-1}(\xi)$, $l = 1,\dots,k$; w_1, w_2, w_3, w_4 are non-negative weights, e.g. $w_1 = w_3 = 0.3$, $w_2 = w_4 = 0.2$. The deterministic objectives (16) are linear even if reference functions L and R are nonlinear.

There exist approaches proposing a substitution of each objective by several deterministic objectives corresponding to extreme values of several ξ-level sets [8, 26].

Finally, let us mention a comparison technique of fuzzy numbers, which is based on the compensation of area determined by the membership functions of two fuzzy numbers being compared. This technique, which has been characterized by Kołodziejczyk [15] and Chanas [4], and then by Roubens [29] and Fortemps and Roubens [13], can be used directly to transform the comparison of fuzzy left- and right-hand side of the constraints, and of the fuzzy objectives and fuzzy goals into non-parametric deterministic equivalents. Although this technique seems intuitive, it has a convincing theoretical foundation.

Indeed, the semantics of fuzzy numbers considered in the MOLP problem with fuzzy coefficients is related to the representation of incomplete or vague states of information under the form of possibility distributions. This view of fuzzy numbers is concordant with the Dempster interpretation of fuzzy numbers as imprecise probability distributions [9]. In this perspective, the comparison of two fuzzy numbers can be substituted by the comparison of their mean values defined consistently with the well-known definition of expectation in probability theory. The idea exploited by Dubois and Prade [12] rely on the mathematical fact that, with respect to a fuzzy number, the possibility measure corresponds to an upper probability distribution, while the necessity measure, to a lower probability distribution of the corresponding random variable. Then it is reasonable to define the mean value of a fuzzy number as a closed interval whose bounds are expectations of upper and lower probability distributions. The comparison of two fuzzy numbers boils then down to the comparison of arithmetic means of these bounds, which is computationally equivalent to the above mentioned technique based on area compensation, as shown in [13].

In consequence of application of all these comparison techniques, the MOLP problem with fuzzy coefficients is transformed to an associate deterministic MOLP problem, as (16), (15), (3) above, which should, preferably, be solved by one of existing interactive procedures (see, e.g., [39]).

4. Flexible MOLP with Fuzzy Coefficients

This problem combines the two semantics of fuzzy sets considered separately in flexible programming and in MOLP with fuzzy coefficients. This means that in addition to fuzzy coefficients in the objective functions and on the both sides of the constraints, the degree of satisfaction of fuzzy constraints and fuzzy goals is considered in fuzzy set terms.

A crucial question which has to be answered while solving a flexible MOLP problem with fuzzy coefficients is how to express the minimal conditions on the satisfaction of fuzzy constraints in deterministic terms.

In most of existing approaches, the minimal conditions on the satisfaction of fuzzy constraints (2) are expressed by one or two deterministic linear constraints which substitute the original fuzzy constraints. To give an idea of these crisp surrogates, let us present them in common terms from the most pessimistic to the most optimistic attitude. We assume the following form of the fuzzy left- and right-hand side of the i-th constraint: $\tilde{\boldsymbol{a}}_i \boldsymbol{x} = \left(\boldsymbol{a}_i^L \boldsymbol{x}, \boldsymbol{a}_i^R \boldsymbol{x}, \alpha_i^L \boldsymbol{x}, \alpha_i^R \boldsymbol{x}\right)_{LR}$ and $\tilde{b}_i = (b_i, 0, \beta_i)_{LR}$.

(a) $\boldsymbol{a}_i^R \boldsymbol{x} + \alpha_i^R \boldsymbol{x}\, R^{-1}(\rho) \le b_i, \quad \rho \in [0,1]$ [2, 37]

(b) $\begin{cases} \boldsymbol{a}_i^R \boldsymbol{x} \le b_i \\ \boldsymbol{a}_i^R \boldsymbol{x} + \alpha_i^R \boldsymbol{x}\, R^{-1}(\varepsilon) \le b_i + \beta_i\, R^{-1}(\varepsilon), \quad \varepsilon \in [0,1] \end{cases}$ [20, 23, 41]

(c) $\boldsymbol{a}_i^R \boldsymbol{x} + \alpha_i^R \boldsymbol{x}\, R^{-1}(\sigma) \le b_i + \beta_i R^{-1}(\sigma), \quad \sigma \in [0,1]$ [3]

(d) $\begin{cases} \boldsymbol{a}_i^L \boldsymbol{x} - b_i \le \alpha_i^L \boldsymbol{x}\, L^{-1}(\tau) + \beta_i\, R^{-1}(\tau), \quad \tau \in (0,1], & \text{optimistic} \\ \boldsymbol{a}_i^R \boldsymbol{x} + \alpha_i^R \boldsymbol{x}\, R^{-1}(\eta) \le b_i + \beta_i\, R^{-1}(\eta), \quad \eta \in [0,1], & \text{pessimistic} \end{cases}$ [7, 32, 33]

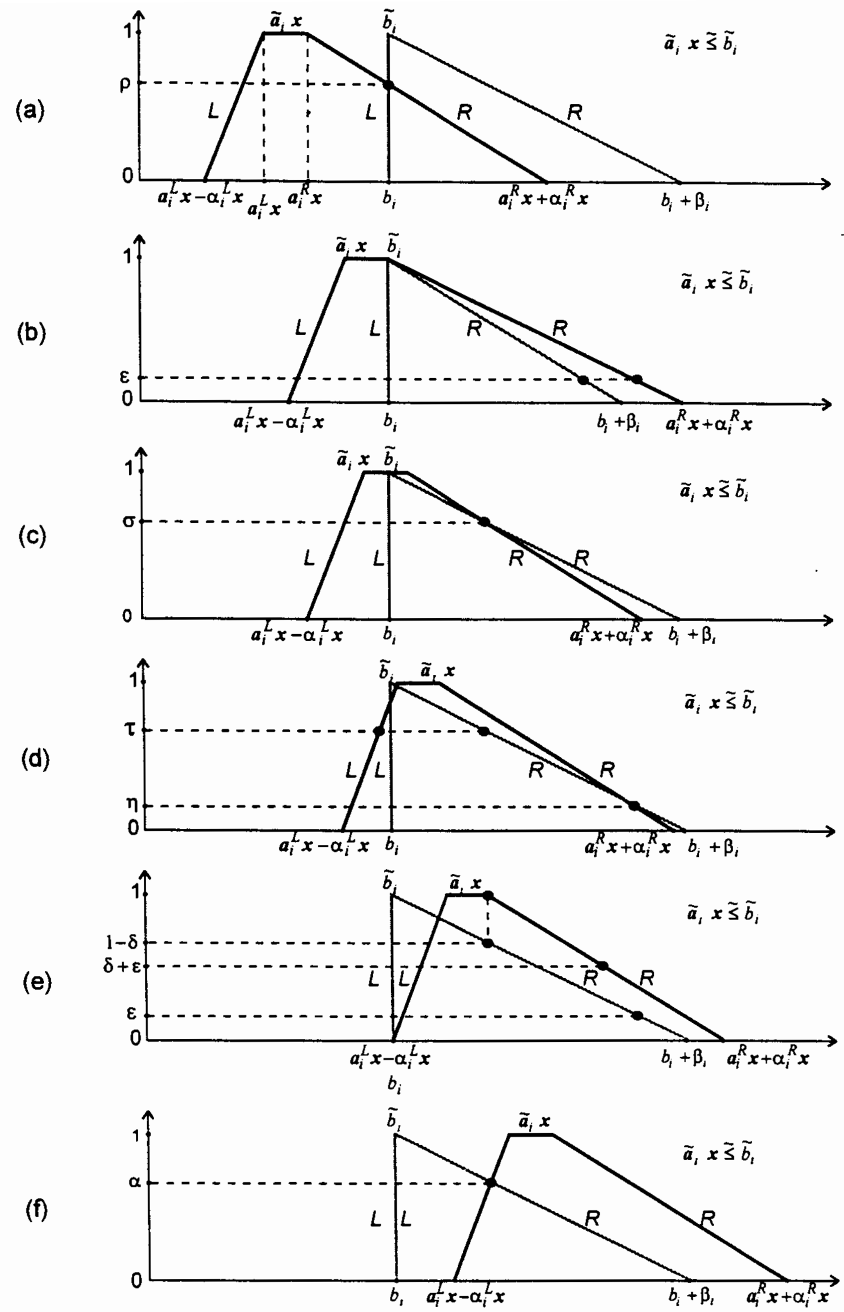

Figure 1: Results of conditions (a) to (f) applied on a common fuzzy constraint

(e) $\begin{cases} \boldsymbol{a}_i^R \boldsymbol{x} \le b_i + \delta\,\beta_i, & \delta + \varepsilon \in [0,1],\ \delta \ge 0,\ \varepsilon \ge 0 \\ \boldsymbol{a}_i^R \boldsymbol{x} + (1-\varepsilon-\delta)\alpha_i^R \boldsymbol{x} \le b_i + (1-\varepsilon)\beta_i \end{cases}$ [21]

(f) $\boldsymbol{a}_i^L \boldsymbol{x} - \alpha_i^L \boldsymbol{x}\, L^{-1}(\alpha) \le b_i + \beta_i\, R^{-1}(\alpha),\ \ \alpha \in [0,1]$ [17, 31]

In all these approaches, the parameters $\alpha, \delta, \varepsilon, \eta, \tau, \rho, \sigma$ can be used by the decision maker to control the degree of satisfaction of fuzzy constraints in an interactive way.

Figure 1 shows results of conditions (a) to (f) applied on a common fuzzy constraint. Although it is the case in Figure 1, the reference functions L and R need not be linear in the above conditions.

Another interpretation of fuzzy constraints has been given in [22]. The i-th fuzzy constraint is replaced by the pessimistic condition proposed in [32] and by a new objective:

$$\boldsymbol{a}_i^R \boldsymbol{x} + \alpha_i^{R\varepsilon} \boldsymbol{x} \le b_i + \beta_i^{\varepsilon} \tag{17}$$

$$\mu_i(\boldsymbol{x}) \to max \tag{18}$$

where membership function $\mu_i(\boldsymbol{x})$ is defined according to (4). More detailed discussion of the interpretation of fuzzy constraints can be found in [28].

If fuzzy goals are specified as L-R fuzzy numbers $\tilde{g}_l = (g_l, 0, v_l)_{LL}$ (l=1,...,k), then the satisfying conditions

$$\tilde{\boldsymbol{c}}_l \boldsymbol{x} \,\tilde{\lesssim}\, \tilde{g}_l, \quad l = 1,\ldots,k \tag{19}$$

can be treated as additional fuzzy constraints. In accordance to the chosen interpretation of the fuzzy inequality relation, (19) can be substituted by one or two crisp inequalities listed above or by (17) and (18). Another proposal has been made in [32, 33]; the degree of satisfaction of fuzzy goals is represented there by the levels of intersection of left reference functions of $\tilde{\boldsymbol{c}}_l \boldsymbol{x}$ with right reference functions of g_l (l=1,...,k):

$$L\left(\left(\boldsymbol{c}_l^L \boldsymbol{x} - g_l\right) / \left(\gamma_l^L \boldsymbol{x} + v_l\right)\right) \to max \quad l = 1,\ldots,k \tag{20}$$

These crisp objectives substitute the fuzzy ones. In the case of linear reference functions L, functions (20) become linear fractional:

$$\left(\boldsymbol{c}_l^L \boldsymbol{x} - g_l\right) / \left(\gamma_l^L \boldsymbol{x} + v_l\right) \to min \quad l = 1,\ldots,k \tag{21}$$

The crisp objectives (21) and the optimistic and pessimistic conditions (d) on the satisfaction of fuzzy constraints have been used in the FLIP method presented in [7, 32, 33, 35]. They constitute an associate deterministic multi-objective linear-fractional programming (MOLFP) problem. In FLIP, the MOLFP problem is solved using an interactive sampling procedure. In each calculation step of this procedure, a sample of non-dominated points (Pareto-optimal solutions) of the MOLFP problem is generated and then shown to the decision maker who is asked to select the one that fits best his/her preferences. If the selected point is not the final compromise, it becomes a central point of a non-dominated region that is sampled in the next calculation step. In this way, the sampled part of the non-dominated set is successively reduced (focusing phenomenon) until the most satisfactory efficient point (compromise solution) is reached. An important advantage of the method presented above is that the only optimization procedure to be used is a linear programming one. Moreover, it has a simple scheme and allows retractions to the points abandoned in previous iterations.

The interaction with the decision maker takes place at two levels: first when fixing the safety parameters and then in the course of the guided generation and evaluation of the non-dominated points of the MOLFP problem.

Let us precise that the fuzzy goals g_l (l=1,...,k) do not influence the set of non-dominated points of the MOLFP problem; they rather play the role of a visual reference than that of a preferential information influencing the set of generated proposals for the compromise solution.

An important feature of any software implementing a fuzzy multiobjective programming method is the presentation of candidate solutions in the interactive process. In the FLIP software, the Pareto-optimal solutions of the MOLFP problem are shown not only numerically but also graphically, in terms of mutual positions of fuzzy numbers corresponding to original objectives and aspiration levels on the one hand, and to left- and right-hand side of original constraints on the other hand [5]. In this way, the decision maker gets quite a complete idea of the quality of each proposed solution.

The quality is evaluated taking into account the following characteristics:

- scores of fuzzy objectives in relation to the goals,
- dispersion of values of the fuzzy objectives due to uncertainty,
- safety of the solution or, using a complementary term, the risk of violation of the constraints.

So, the definition of the best compromise involves not only the scores on particular objectives but also the safety of the corresponding solution. It is possible due to *visual interaction* that needs graphical display of objectives and constraints for any analyzed solution. The comparison of fuzzy left- and right-hand side of the constraints, as well as evaluation of dispersion of the values of objectives, is practically infeasible on the basis of numerals only. The graphical presentation of proposed solutions is not only a "user's friendly" interface but the best way for a complete characterization of these solutions.

There exists an implementation of FLIP in Visual Basic in the MS-Excel environment; it allows a user to define all safety parameters and the parameter p of the Yager's formula (13) for the aggregation of fuzzy objectives and of fuzzy left-hand sides of fuzzy constraints. The candidates for the best compromise solution are displayed there both numerically and graphically.

5. Conclusions

Fuzzy multi-objective linear programming methods have often been proposed in view of specific applications (see, e.g., [5, 16, 28, 32, 36, 40]). This means that the many proposals described in this paper are based on different assumptions that are verified in different practical situations. The choice of a procedure for an actual decision problem should take into account these assumptions. In any case, the interactive process should enable the best use of the decision maker's knowledge of the problem.

Fuzzy multi-objective linear programming can also be seen as a tool for an interactive *robustness analysis* of MOLP problems. It gives an insight into sensitivity of proposed solutions on changes of particular coefficients within some intervals and on changes of preferences as to degrees of satisfaction of the constraints.

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BOOK REVIEW

Title : Programmation linéaire
Author : J. Teghem

Publisher : Editions de l'Université Libre de Bruxelles (Bruxelles) et Editions Ellipses (Paris), Collection "Statistiques et Mathématiques Appliquées", 1996, 374 p.

ISBN: 2-8004-1150-3

Was it really necessary to write another book on such a classical subject as Linear Programming? A simple look at the present work readily shows that the answer is clearly "Yes!" since there is no other book of that kind. The new book by Jacques Teghem offers an up-to-date presentation of Linear Programming in French and is ideal as a basis for teaching this material at various levels.

In view of a first course in LP, the author introduces the classical simplex algorithm, duality and the dual simplex algorithm as well as the basics of branch and bound methods for integer programming. More original, the opportunity is offered to the reader to obtain at very moderate price a student version of the LP-MIP commercial package of OM Partners; extensive instructions for using the package are provided as well as a number of illustrative examples which can either be computed by hand or solved by using the software.

The second level topics include
- the primal-dual algorithm and its application to solving the assignment problem and the transportation problem;
- the handling of boundary constraints on the variables;
- the resolution of linear parametric programs.

Again, an original part consists in two chapters dealing respectively with stochastic and fuzzy linear programming and with multicriteria linear programming both subjects in which the author has been and is very active. Note in particular that the presentation of fuzzy linear programming is self-contained since the relevant concepts of fuzzy sets theory are introduced in a very readable manner. For multicriteria linear programming, a number of interactive methods are outlined; the chapter closes with a description of two methods which encompass both the multiobjective and stochastic or fuzzy aspects; the so-called STRANGE method, due to the author and his collaborators, and the FLIP method due to R. Slowinski have both been developed for solving practical planning problems respectively in the domains of energy production and in agriculture. Uncertainty and imprecision of the data on one hand and the coexistence of several irreconcilable objectives on the other hand are indeed basic features of those important planning problems.

Finally, at an advanced level, the revised simplex algorithm and the Dantzig-Wolfe decomposition are presented both based on a column generation technique and well-suited for large scale problems. In chapter XII, the basics of problem complexity theory are provided which allows to understand the interest of the Kachyan algorithm (ellipsoïd algorithm) and of the Karmarkar algorithm; while the former only has had theoretical interest in showing that linear programs were solvable in polynomial time, the latter is also powerful in practice.

This panorama of linear programming certainly contains all the material usually covered in courses at undergraduate and graduate levels. Since it is written with great paedagogical sense and fully illustrated with numerical examples and applications, there is no doubt that this book can be used with great profit by students (engineers, economists, mathematicians, ...) as well as by their instructors; those who received an elementary course on LP during their university studies can also find in this book a convenient and pleasant way for improving and updating their knowledge of the field.

Marc Pirlot.