UNRELIABLE RETRIAL QUEUES DUE TO SERVICE INTERRUPTIONS ARISING FROM FACSIMILE NETWORKS

J. R. Artalejo and A. Gómez-Corral

Facultad de Matematicas
Departamento de Estadistica e I.O.
Universidad Complutense de Madrid
Madrid 28040, Spain

Abstract

Queueing systems with retrials arise naturally in telecommunications and computer systems. The increasing interest on this topic is mainly explained by the development of new facilities in telecommunication technology such as "repeat last number", "ring back when free", etc. In this paper we concentrate our attention in the reliability of the system when the server is subject to interruptions. We consider a retrial queueing system with service interruptions arising from facsimile networks. We first investigate the joint distribution of the channel state and the number of customers in the retrial group in the steady state. We show that the queue length characteristics reduce to closed form expressions for the case of exponential service times. Using some results from the theory of regenerative processes, we also obtain some other important performance measures.

Keywords: facsimile transmission, queue length distribution, reliability, regenerative processes, retrial queues, stable recursive schemes

1 Introduction

Over recent years it has been a rapid growth in the literature on Retrial Queueing Systems. This growth is mainly motivated by the applications to many telecommunication and computer systems. Interested readers can find a comprehensive survey in Yang and Templeton [15], Falin [8] and Falin and Templeton [10]. Additional information about unreliable retrial queues can be found in Aissani [1], Anisimov and Atadzhanov [4] and Kulkarni and Choi [12].

In this paper, we study an M/G/1 queue with repeated attempts in which the server operates under the linear retrial policy analyzed by Artalejo and Gómez-Corral [6]. In addition, we assume that the server is subject to interruptions. The customer whose service time is interrupted has to either leave the system or join again the retrial group.

Our model is related to some variants of the main retrial queue of type M/G/1. For example, the service interruptions can be view as a special type of breakdown of the server (see Aissani [1,2], Aissani and Artalejo [3], Artalejo [5] and their references) where the server is restarted instantaneously. Our model falls also into the category of retrial queues with feedback (see Choi and Kulkarni [7]). Here, we would like to pointed out that our specific formulation allows us to study the queueing model in more depth.

The major motivation for our model comes from auto-repeat facilities for the transmission of messages in facsimile networks. Consider a communication enterprise which reserves a specific facsimile machine for sending messages to outside destinations. If the channel is free when a user arrives and demands to send a message, then the transmission starts. On the other hand, any message finding the channel busy must be stored in a buffer, but some time later the demand is reinitiated. In any case, after occupation of the free channel a message transmission starts. If the transmission is not concluded for reasons such as a blocking in an external link or a transmission error, then the message leaves the channel and joins the buffer. It should be pointed out that the facsimile equipment remains operative and the service interruptions must be explained only in terms of external factors. This is the main difference with queueing systems subject to classical breakdowns. The aim of this paper is to investigate the implications of service interruptions in the consequent repeat-attempt behaviour. The analysis of more complex models (where the facsimile machine acts simultaneously as receiver and transmitter, or alternates 'fax' and 'phone' periods) may be the subject matter of any forthcoming work.

The rest of the paper is organized as follows. In Section 2, we describe the mathematical model. The study of the system state in steady state is carried out in Section 3. This analysis includes the recursive computation of the limiting probabilities, z-transforms and study of the model at Markovian level. Finally, other important performance characteristics are derived in Section 4.

2 Model description

We consider a single server queueing system at which primary customers arrive according to a Poisson process with rate λ. Any arriving customer who finds the server busy upon arrival leaves the service area and joins the retrial group. The control discipline to access from the retrial group to the server is governed by an exponential law with linear intensity $\alpha(1-\delta_{0j})+j\mu$, when the number of units at the retrial group is $j \in \mathbb{N}$, where δ_{ab} denotes Kronecker's delta. In the case $\alpha = 0$ and $\mu > 0$ the retrial intensity becomes the classical retrial discipline (see Yang and Templeton [15] and Falin [8]). Alternatively, when $\mu = 0$ and $\alpha > 0$ we obtain the constant retrial discipline (see Martin and Artalejo [13] and its references). The service times are general with probability distribution function $B(t)$ ($B(0) = 0$), conditional completion rate $\eta(x)$ and Laplace-Stieltjes transform $\beta(s)$. In addition, we assume that the server is subject to interruptions. The customer whose service is interrupted has two possibilities: either to return to the retrial group and try his luck later, or leave the system. We incorporate both choices by introducing two more exponential laws, i.e., the server fails at an exponential rate θ_1 (respectively, θ_2) and then the customer leaves the system (respectively, returns to the retrial group). Alternatively, we can think only in one exponential law with rate θ and a recovery factor, defined as the probability that the interrupted customer rejoins the retrial group. We finally assume that the input flow of primary arrivals, intervals between repeated attempts, interruptions and service times are mutually independent.

Note that the state of the system at time t can be described by the process $X(t) = (C(t), N(t), \xi(t))$, where $C(t)$ takes values on $\{0,1\}$ according as the server is idle or busy. $N(t)$ represents the number of customers in the retrial group at time t. If $C(t) = 1$, then $\xi(t)$ denotes the elapsed time of the customer being served. The transitions among states are illustrated in Figure 1.

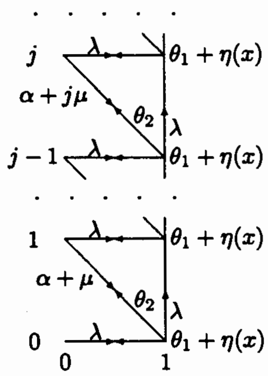

Figure 1. Transitions among states

The model described above has applications in the design of the transmission of messages in facsimile machines. The customer in the retrial group, the distribution of the successive repeated attempts and the service time distribution in the queueing terminology correspond respectively to the blocked/interrupted messages stored in the buffer, retransmission protocol and transmission time. In practice, different transmission errors yield either the return of the interrupted message to the buffer or a definitive deletion.

In what follows, we assume that one of the following conditions is fulfilled:

i) If $\alpha > 0$ and $\mu = 0$, then $\rho\alpha^{-1}(\lambda + \alpha + \theta_2) < 1$,
ii) If $\alpha \geq 0$ and $\mu > 0$, then $\rho < 1$,

where $\rho = \lambda(1 - \beta(\theta_1 + \theta_2))(\theta_1 + \theta_2\beta(\theta_1 + \theta_2))^{-1}$.

By using classical criteria based on mean drifts, it can be easily proved that the above conditions determine the ergodicity of the embedded Markov chain at the sequence of epochs which either a service completion or a service interruption occurs. Thus, in the rest of the paper we assume these conditions to guarantee that the limiting probabilities of $(C(t), N(t))$ exist and are positive.

3 Analysis of the limiting distribution

Now, we study the limit behaviour of the process $(C(t), N(t))$ as $t \to \infty$. Our first goal is to develop a stable recursive scheme for the limiting probabilities

$$P_{ij} = \lim_{t\to\infty} P\{(C(t), N(t)) = (i,j)\}, \qquad (i,j) \in E = \{0,1\} \times \mathbb{N}. \tag{3.1}$$

Our derivation is based on a versatile regenerative approach (see Tijms [14]) which was also the key to compute the steady state distribution in other retrial queues (see de Kok [11] and Artalejo [5]). Let a regeneration cycle be the time elapsed between two consecutive primary arrivals finding the system empty. The process $(C(t), N(t))$ regenerates itself at these epochs. We define an *extended service time* as the up time when the server begins a service until the next completion or interruption, i.e., it is the minimum among the service time and the exponential laws governing both types of interruptions. We also define some random variables:

T: the length of a cycle,

T_{ij}: the amount of time in a cycle during which the system state is (i,j),

N_j^s: the number of extended service completions in a cycle, concluding with a service completion, at which j customers are left behind in the retrial group,

N_j^k: the number of extended service completions in a cycle, concluding with an interruption of type $k \in \{1,2\}$, at which j customers are left behind in the retrial group.

By the theory of regenerative processes, we can express the limiting probabilities (3.1) as

$$P_{ij} = \frac{E[T_{ij}]}{E[T]}, \qquad (i,j) \in E. \tag{3.2}$$

We now consider the following balance equations:

$$(\lambda+\alpha(1-\delta_{0j})+j\mu)E[T_{0j}] = E[N_j^s]+\theta_1 E[T_{1j}]+(1-\delta_{0j})\theta_2 E[T_{1,j-1}], \quad j\ge 0, \qquad (3.3)$$

$$(\alpha+(j+1)\mu)E[T_{0,j+1}] = (\lambda+\theta_2)E[T_{1j}], \quad j\ge 0. \qquad (3.4)$$

Equations (3.3) and (3.4) can be obtained by equating the flow rate into and the flow rate out of $(0,j)$ and $\{(i,m):\ i\in\{0,1\},\ j\ge m\ge 0\}$, respectively.

Now we divide both sides of (3.4) by $E[T]$ and invoke to (3.2). Then, we have

$$(\alpha+(j+1)\mu)P_{0,j+1} = (\lambda+\theta_2)P_{1j}, \qquad j\ge 0. \qquad (3.5)$$

In view of (3.5) our problem is reduced to the computation of the sequence $\{P_{1j};\ j\ge 0\}$.

To find a relation among $E[T_{1j}]$, $E[N_j^s]$ and $E[N_j^k]$, we introduce the auxiliary quantity

A_{kj}: the expected amount of time that during an extended service j customers are in the retrial group, given that the previous extended service left k customers in the retrial group.

Now an easy application of Wald's theorem allows us to get

$$E[T_{1j}] = \sum_{k=0}^{j+1}\left(E[N_k^s]+E[N_k^1]+E[N_k^2]\right)A_{kj}, \quad j\ge 0. \qquad (3.6)$$

On the other hand, since the interruptions are exponentially distributed, we find that

$$E[N_k^1] = \theta_1 E[T_{1k}], \quad k\ge 0, \qquad (3.7)$$

$$E[N_k^2] = \theta_2 E[T_{1,k-1}], \quad k\ge 1. \qquad (3.8)$$

By combining (3.3), (3.4), (3.6)-(3.8), it follows that

$$E[T_{1j}] = A_{0j} + (\lambda+\theta_2)\sum_{k=1}^{j+1}\left(1+\frac{\lambda}{\alpha+k\mu}\right)A_{kj}E[T_{1,k-1}], \quad j\ge 0. \qquad (3.9)$$

Observe that $E[T_{00}] = 1/\lambda$. Thus, we find the useful relation $E[T] = 1/(\lambda P_{00})$. Dividing both sides of (3.9) by $E[T]$, we find the recurrence relation

$$P_{1j} = \lambda P_{00}A_{0j} + (\lambda+\theta_2)\sum_{k=1}^{j+1}\left(1+\frac{\lambda}{\alpha+k\mu}\right)A_{kj}P_{1,k-1}, \quad j\ge 0. \qquad (3.10)$$

Hence we can compute $\{P_{1j}; j\ge 0\}$, in terms of P_{00}, by a stable recursive scheme once we have evaluated the quantities A_{kj}. Finally, we can get P_{00} by using the normalizing condition $\sum_{(i,j)\in E} P_{ij} = 1$. It remains to specify the coefficients A_{kj}. To that end, we define

B_{kj}: the expected amount of time that during an extended service j customers are in the retrial group, given that is started with k customers.

Both auxiliary quantities are connected by the following relationships:

$$A_{j+1,j} = \frac{\alpha + (j+1)\mu}{\lambda + \alpha + (j+1)\mu} B_{jj}, \qquad j \geq 0, \tag{3.11}$$

$$A_{kj} = \frac{\alpha(1-\delta_{0k}) + k\mu}{\lambda + \alpha(1-\delta_{0k}) + k\mu} B_{k-1,j} + \frac{\lambda}{\lambda + \alpha(1-\delta_{0k}) + k\mu} B_{kj}, \;\; j \geq k \geq 0. \tag{3.12}$$

Observe that an infinitesimal interval $(t, t+\Delta t)$ contributes to B_{kj} if: *i)* the service time has not been completed before time t (with probability $1 - B(t)$), *ii)* a service interruption did not occur before time t (with probability $exp\{-(\theta_1+\theta_2)t\}$), and *iii)* $j-k$ primary customers arrive to the system in the interval $(0,t)$.

Then, we have

$$B_{kj} = \int_0^\infty e^{-\lambda t} \frac{(\lambda t)^{j-k}}{(j-k)!} (1 - B(t)) e^{-(\theta_1+\theta_2)t} dt, \qquad j \geq k \geq 0. \tag{3.13}$$

Note that (3.10) can be reexpressed as follows:

$$(1-(\lambda+\theta_2)a_0)P_{1j} = \lambda P_{00} a_j + (1-\delta_{0j})(\lambda+\theta_2) \sum_{k=1}^{j} \left(a_{j-k+1} + \frac{\lambda}{\alpha + k\mu} a_{j-k} \right) P_{1,k-1}, \;\; j \geq 0, \tag{3.14}$$

where

$$a_j = \int_0^\infty e^{-(\lambda+\theta_1+\theta_2)t} \frac{(\lambda t)^j}{j!} (1 - B(t)) dt, \qquad j \geq 0. \tag{3.15}$$

The integral in (3.15) can be reduced to finite sums for the case of the most usual service time distributions.

For the sake of completeness, we next study the partial generating functions $P_i(z) = \sum_{j=0}^{\infty} P_{ij} z^j$, $i \in \{0,1\}$.

Theorem 3.1. *The partial generating functions $P_i(z)$ are given by:*

$$P_1(z) = \frac{\lambda(1 - \beta(\lambda - \lambda z + \theta_1 + \theta_2))}{(\lambda + \theta_2)\beta(\lambda - \lambda z + \theta_1 + \theta_2) + \theta_1 - \lambda z} P_0(z). \tag{3.16}$$

If $\alpha \geq 0$ and $\mu > 0$, then

$$P_0(z) = z^{-\alpha/\mu} \left((1-\rho) \exp\left\{ \int_z^1 \frac{\lambda(\lambda+\theta_2)(1 - \beta(\lambda - \lambda t + \theta_1 + \theta_2))}{\mu(\lambda t - \theta_1 - (\lambda+\theta_2)\beta(\lambda - \lambda t + \theta_1 + \theta_2))} dt \right\} \right.$$

$$-\alpha\mu^{-1}P_{00}\int_z^1 t^{\alpha/\mu-1}exp\left\{\int_z^t \frac{\lambda(\lambda+\theta_2)(1-\beta(\lambda-\lambda u+\theta_1+\theta_2))}{\mu(\lambda u-\theta_1-(\lambda+\theta_2)\beta(\lambda-\lambda u+\theta_1+\theta_2))}du\right\}dt\right), \quad (3.17)$$

where

$$P_{00}=\begin{cases}(1-\rho)\ exp\left\{\int_0^1 \frac{\lambda(\lambda+\theta_2)(1-\beta(\lambda-\lambda t+\theta_1+\theta_2))}{\mu(\lambda t-\theta_1-(\lambda+\theta_2)\beta(\lambda-\lambda t+\theta_1+\theta_2))}dt\right\}, & \text{for } \alpha=0,\\ (1-\rho)\left(\frac{\alpha}{\mu}\int_0^1 t^{\alpha/\mu-1}exp\left\{\int_1^t \frac{\lambda(\lambda+\theta_2)(1-\beta(\lambda-\lambda u+\theta_1+\theta_2))}{\mu(\lambda u-\theta_1-(\lambda+\theta_2)\beta(\lambda-\lambda u+\theta_1+\theta_2))}du\right\}dt\right)^{-1}, & \text{for } \alpha>0.\end{cases} \quad (3.18)$$

If $\alpha>0$ *and* $\mu=0$, *then*

$$P_0(z)=P_{00}\left(1+\frac{\lambda z(\lambda+\theta_2)(1-\beta(\lambda-\lambda z+\theta_1+\theta_2))}{\alpha(\lambda z-\theta_1-(\lambda+\theta_2)\beta(\lambda-\lambda z+\theta_1+\theta_2))}\right)^{-1}, \quad (3.19)$$

where

$$P_{00}=\frac{\alpha\theta_1+(\lambda+\alpha)(\lambda+\theta_2)\beta(\theta_1+\theta_2)-\lambda(\lambda+\alpha+\theta_2)}{\alpha(\theta_1+\theta_2\beta(\theta_1+\theta_2))}. \quad (3.20)$$

Proof. From (3.5) and (3.14) we get an alternative recurrence relation involving the probabilities $\{P_{0j};\, j\geq 0\}$:

$$(1-(\lambda+\theta_2)a_0)(\lambda+\theta_2)^{-1}(\alpha+(j+1)\mu)P_{0,j+1}=\lambda P_{00}a_j$$
$$+(1-\delta_{0j})\sum_{k=1}^{j}(\lambda a_{j-k}+(\alpha+k\mu)a_{j-k+1})P_{0k}, \qquad j\geq 0. \quad (3.21)$$

Then, taking transforms over (3.21) we find after some rearrangements that

$$P_0'(z)\mu z(1-(\lambda+\theta_2)a(z))+P_0(z)(\alpha(1-(\lambda+\theta_2)a(z))-\lambda(\lambda+\theta_2)za(z))$$
$$=\alpha P_{00}(1-(\lambda+\theta_2)a(z)), \quad (3.22)$$

where

$$a(z)=\sum_{j=0}^{\infty}a_jz^j=\frac{1-\beta(\lambda-\lambda z+\theta_1+\theta_2)}{\lambda-\lambda z+\theta_1+\theta_2}. \quad (3.23)$$

Putting $\mu=0$ in (3.22) we obtain readily (3.19). The discussion of the case $\mu>0$ leads to the differential equation

$$P_0'(z)+\left(\frac{\alpha}{\mu z}+\frac{\lambda(\lambda+\theta_2)(1-\beta(\lambda-\lambda z+\theta_1+\theta_2))}{\mu(\lambda z-\theta_1-(\lambda+\theta_2)\beta(\lambda-\lambda z+\theta_1+\theta_2))}\right)P_0(z)=\frac{\alpha}{\mu z}P_{00}. \quad (3.24)$$

Taking transforms in (3.5), we find that

$$\alpha z^{-1}(P_0(z)-P_{00})+\mu P_0'(z)=(\lambda+\theta_2)P_1(z). \quad (3.25)$$

We now combine (3.24) and (3.25) to get the expression (3.16). The solution of the equation (3.24) is standard and thus omitted. □

In Theorem 3.1, we have omitted the derivation of the factorial moments defined by $M_k^i = \sum_{j=0}^{\infty} \binom{j}{k} k! P_{ij}$, $i \in \{0,1\}$, $k \geq 0$. They can be obtained upon suitable differentiation over the corresponding transforms after tedious algebraic calculations.

Because of the complexity of the formulas for generating functions given in Theorem 3.1, it would be interesting to obtain closed form expressions for the pure Markovian model, i.e., $B(t) = 1 - e^{-\nu t}$, $t \geq 0$. We summarize in the following theorem some explicit results for this case.

Theorem 3.2. *If $\alpha \geq 0$ and $\mu > 0$, then the stationary distribution of the process $(C(t), N(t))$ is given by:*

$$P_{0j} = \frac{\lambda + \theta_2}{\mu} \frac{\left(1 + \frac{\lambda+\alpha+\theta_2}{\mu}\right)_{j-1}}{\left(1 + \frac{\alpha}{\mu}\right)_j} \rho^j P_{00}, \quad j \geq 1, \tag{3.26}$$

$$P_{1j} = \frac{\left(1 + \frac{\lambda+\alpha+\theta_2}{\mu}\right)_{j}}{\left(1 + \frac{\alpha}{\mu}\right)_j} \rho^j P_{10}, \quad j \geq 1, \tag{3.27}$$

$$P_{00} = \rho^{-1} P_{10}, \qquad P_{10} = \rho \left(F\left(1, 1 + \frac{\lambda + \alpha + \theta_2}{\mu}; 1 + \frac{\alpha}{\mu}; \rho\right)\right)^{-1}, \tag{3.28}$$

where $\rho = \lambda(\nu + \theta_1)^{-1}$, $(x)_n$ is the Pochhammer symbol and F denotes the hypergeometric series defined as follows

$$(x)_n = \begin{cases} 1, & \text{for } n = 0, \\ x(x+1)...(x+n-1), & \text{for } n \geq 1, \end{cases} \tag{3.29}$$

$$F(a, b; c; z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!}.$$

The k-th partial factorial moments are given by:

$$M_k^0 = P_{00} \frac{\lambda + \theta_2}{\lambda + \alpha + \theta_2} k! \rho^k \frac{\left(\frac{\lambda+\alpha+\theta_2}{\mu}\right)_k}{\left(1 + \frac{\alpha}{\mu}\right)_k} F\left(1 + k, \frac{\lambda + \alpha + \theta_2}{\mu} + k; \frac{\alpha}{\mu} + 1 + k; \rho\right), \quad k \geq 1, \tag{3.30}$$

$$M_k^1 = P_{10} k! \rho^k \frac{\left(1 + \frac{\lambda+\alpha+\theta_2}{\mu}\right)_k}{\left(1 + \frac{\alpha}{\mu}\right)_k} F\left(1 + k, \frac{\lambda + \alpha + \theta_2}{\mu} + 1 + k; \frac{\alpha}{\mu} + 1 + k; \rho\right) \quad k \geq 1, \tag{3.31}$$

$$M_0^0 = 1 - \rho, \qquad M_0^1 = \rho. \tag{3.32}$$

If $\alpha > 0$ and $\mu = 0$, then the stationary distribution of the process $(C(t), N(t))$ is given by:

$$P_{0j} = \frac{\lambda + \theta_2}{\alpha}\gamma^{j-1}\rho P_{00}, \qquad P_{1j} = \gamma^j P_{10}, \qquad j \geq 1, \tag{3.33}$$

$$P_{00} = \rho^{-1}P_{10}, \qquad P_{10} = \frac{\lambda(\alpha(\nu + \theta_1) - \lambda(\lambda + \alpha + \theta_2))}{\alpha(\nu + \theta_1)^2}, \tag{3.34}$$

and its corresponding k-th partial factorial moments are given by:

$$M_k^0 = k!\frac{\lambda + \theta_2}{\lambda + \alpha + \theta_2}\left(\frac{\gamma}{1-\gamma}\right)^k, \qquad M_k^1 = k!\rho\left(\frac{\gamma}{1-\gamma}\right)^k, \qquad k \geq 1, \tag{3.35}$$

$$M_0^0 = 1 - \rho, \qquad M_0^1 = \rho, \tag{3.36}$$

where $\gamma = \alpha^{-1}(\lambda + \alpha + \theta_2)\rho$.

Proof. First we consider the Kolmogorov equations for the probabilities $\{P_{ij};\ (i,j) \in E\}$:

$$(\lambda + \alpha(1 - \delta_{0j}) + j\mu)P_{0j} = \theta_2(1 - \delta_{0j})P_{1,j-1} + (\nu + \theta_1)P_{1j}, \quad j \geq 0, \tag{3.37}$$

$$(\lambda + \nu + \theta_1 + \theta_2)P_{1j} = \lambda P_{0j} + (\alpha + (j+1)\mu)P_{0,j+1} + \lambda(1 - \delta_{0j})P_{1,j-1}, \quad j \geq 0. \tag{3.38}$$

In fact, the system (3.37)-(3.38) can be simplified by substituting (3.38) by (3.5). Equations (3.26)-(3.28) and (3.33)-(3.34) follow after some algebraic manipulations.

The partial generating functions can be obtained from the stationary probabilities P_{ij}. Of course, a suitable substitution on the results given in Theorem 3.1 also leads to the same quantities. In any case, it is easy to get the following expressions for the case $\mu > 0$.

$$P_0(z) = P_{00}\left(\frac{\alpha}{\lambda + \alpha + \theta_2} + \frac{\lambda + \theta_2}{\lambda + \alpha + \theta_2}F\left(1, \frac{\lambda + \alpha + \theta_2}{\mu}; 1 + \frac{\alpha}{\mu}; \rho z\right)\right), \tag{3.39}$$

$$P_1(z) = P_{10}F\left(1, 1 + \frac{\lambda + \alpha + \theta_2}{\mu}; 1 + \frac{\alpha}{\mu}; \rho z\right). \tag{3.40}$$

Since $(1+z)^n = \sum_{k=0}^{n}\binom{n}{k}z^k$, we find that

$$P_i(1+z) = \sum_{k=0}^{\infty} M_k^i \frac{z^k}{k!}, \qquad i \in \{0, 1\}. \tag{3.41}$$

Thus, the expressions (3.30)-(3.32) for M_k^i can be obtained by a direct identification for the coefficients of the series $P_i(1+z)$.

Putting $\mu = 0$ over (3.30) and (3.31) we finally get the expressions (3.35) for the case $\mu = 0$ and $\alpha > 0$. □

4 Other performance characteristics

In previous section we gave a detailed study of the limiting distribution of the system state. The queue length process is the most extensively studied characteristic of any retrial queue. The analysis of other performance measures such as the busy period or the waiting time process leads to extremely cumbersome formulas. Another area of work is the case of finite input stream (see Falin and Artalejo [9]). In the context of the facsimile network, this situation occurs if the population having access to the facsimilie machine has a moderate size. Then the input can be described in terms of the so-called "quasi random input" rather than in terms of a Poisson process.

To conclude we obtain some other measures of effectiveness in the following.

Theorem 4.1. *Let us assume the ergodicity condition given in Section 2. Then, we have*

i) The mean length of a regeneration cycle is

$$E[T] = \frac{1}{\lambda P_{00}}. \tag{4.1}$$

ii) The mean number of extended service completions in a regeneration cycle is

$$E[N] = \frac{\theta_1 + \theta_2}{P_{00}(\theta_1 + \theta_2 \beta(\theta_1 + \theta_2))}. \tag{4.2}$$

iii) The mean number of customers served and the mean number of service interruptions of type k, $k \in \{1,2\}$, are respectively given by:

$$E[N^s] = \frac{(\theta_1 + \theta_2)\beta(\theta_1 + \theta_2)}{P_{00}(\theta_1 + \theta_2 \beta(\theta_1 + \theta_2))}, \tag{4.3}$$

$$E[N^k] = \frac{\theta_k(1 - \beta(\theta_1 + \theta_2))}{P_{00}(\theta_1 + \theta_2 \beta(\theta_1 + \theta_2))}, \quad k \in \{1,2\}. \tag{4.4}$$

Proof. Expression (4.1) follows directly from (3.2). A new appeal to (3.2) allows us to express the expected amount of time in a cycle during which the server is busy as

$$E\left[\sum_{j=0}^{\infty} T_{1j}\right] = \rho E[T] = \frac{\rho}{\lambda P_{00}}. \tag{4.5}$$

Now expression (4.2) follows from Wald's theorem and (4.5).

Finally, we observe that the conditional distribution of N^k, given that $\{N = i\}$, follows a Bernoulli law of i trials with probability $\theta_k(\theta_1+\theta_2)^{-1}(1-\beta(\theta_1+\theta_2))$ of success. Thus, we can easily prove the validity of (4.4). $E[N^s] = E[N] - E[N^1] - E[N^2]$, so we get (4.3). □

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