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Geometry of Left-continuous Triangular Norms with Strong Induced Negations

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Abstract

The purpose of this paper is to make the first step toward the understanding of the structure of left-continuous triangular norms with strong induced negations. For two-placed functions on $[0, 1]$ two properties are introduced: The rotation invariance property and the self quasi-inverse property. It is proved that these properties are characteristic for the class left-continuous triangular norms with strong induced negations. The two properties turn out to be equivalent on the class of symmetric, non-decreasing two-place functions on $[0, 1]$, that is, such a function admits the rotation invariance property if and only if it admits the self quasi-inverse property. These properties have equivalent geometrical counterparts which are investigated, explained in detail and examples are given. These geometrical counterparts can be represented in 3 and 2 dimensions.

Keywords: triangular norm, residual implication, induced negation, involution, left-continuity, rotation, reflection.

1 Introduction

Triangular norms (t-norms for short) are associative functions that play a basic role in several disciplines of mathematics. In the theory of probabilistic (statistical) metric spaces ([9]), they model the "triangle inequality" of a probabilistic metric space, where the distance of two objects is described by a probability distribution instead of a real number. In fuzzy set theory, together with their dual operators – the triangular conorms – they model the intersection and union of fuzzy subsets, respectively. In fuzzy logic, t-norms and t-conorms model 'AND' and 'OR', the logical conjunction and disjunction. In the field of decision analysis, fuzzy preference modeling ([1],[4]) – due to its strong correspondence with fuzzy logic – uses t-norms and t-conorms as well. T-norms are applied in fuzzy control and so on.

The condition of left-continuity is a frequently cited property and plays a central role in all the fields that use t-norms. Left-continuous t-norms with strong induced negations are even more relevant. In spite of their significance, the knowledge about left-continuous t-norms is rather poor at present; there are no results in the literature where left-continuous t-norms stand as the focus of interest. Moreover, until recently there were no known examples left-continuous t-norms, except for the standard class of continuous t-norms. Recently, two basic families of left-continuous (but not continuous) t-norms with strong induced negations have been discovered.

The characterization of associativity is one of the oldest open problems in mathematics. Classification results of solutions of the associativity equation

$$f(f(x,y),z) = f(x,f(y,z))$$

without assuming further strong conditions are not known yet and we are very far from reaching this level of understanding. We understand well symmetry (commutativity) of functions (operations) as some invariance of their graphs with respect to a certain reflection. But up to the author's knowledge, there exists no similar result for associativity. Of course, associativity together with commutativity means some symmetry property in a four-dimensional space, but since our way of looking at things is limited to three dimensions this four-dimensional interpretation doesn't really support our mathematical intuitions and is not very helpful at formulating mathematical conjectures.

In order to fill in the gap between the particular importance of left-continuous t-norms and the pure knowledge about them on the one hand, and in order to understand associativity better on the other hand, we discuss here the geometrical properties of left-continuous t-norms with strong induced negations. We associate two clear and simple geometrical meanings to the associativity of such two-placed functions: we introduce the rotation invariance property and the self quasi-inverse property and explain their geometrical content. The first property can be represented in three dimensions, the second one in two dimensions. Hence both properties are easy to understand with our 'maximum three-dimensional brain'. Moreover, these properties turn out to be equivalent on the class of symmetric, non-decreasing two-place functions on $[0,1]$, that is, if such a function admits one of those properties then it admits the other one too. We prove that each left-continuous t-norm with strong induced negation has those properties. Up to the author's knowledge, this is the first time that associativity (together with other conditions) is represented somehow in such an understandable way in two or three dimensions.

Moreover, the understanding of the geometrical content of the self quasi-inverse and the rotation invariance properties will be the basis for understanding the structure of left-continuous t-norms with strong induced negations (see e.g. [6]).

Finally, we remark that results of the present paper have been presented in international conferences (first in [7]).

2 Basic definitions

First, we repeat the essential definitions.

Definition A A *triangular norm* (t-norm for short) is a function $T : [0,1]^2 \to [0,1]$ such that for all $x, y, z \in [0,1]$ the following four axioms (T1)–(T4) are satisfied:

(T1) Symmetry	$T(x,y) = T(y,x)$	
(T2) Associativity	$T(x, T(y,z)) = T(T(x,y), z)$	
(T3) Monotonicity	$T(x,y) \le T(x,z)$	whenever $y \le z$
(T4) Boundary condition	$T(x,1) = x$	
(T4') Boundary condition	$T(0,x) = 0.$	

It is immediate to see that (T3) and (T4) imply (T4'). A t-norm is said to be *continuous* resp. *left-continuous* if it is continuous resp. left-continuous as a two-place function.

Definition B A *negation* N is a non-increasing function on $[0,1]$ with boundary conditions $N(0) = 1$ and $N(1) = 0$. Such a negation is called *strong* if N is an involution, that is, $N(N(x)) = x$ holds for all $x \in [0,1]$.

A negation is strong if and only if its graph is invariant w.r.t. the reflection at the line $y = x$. A strong negation is automatically a strictly decreasing and continuous function.

Definition C Let $T : [0,1]^2 \to [0,1]$ be a function satisfying (T1) and (T3). The implication function I_T generated by T is given by

$$I_T(x,y) = \sup\{t \in [0,1] \mid T(x,t) \le y\}.$$

If T is left-continuous then I_T is called the *residual implication* generated by T.

Define

$$N_T(x) = I_T(x,0),$$

for $x \in [0,1]$. It is easy to see that left-continuity of T implies

$$T(x,y) = 0 \;\Leftrightarrow\; x \le N_T(y), \tag{1}$$

and left-continuity of T is equivalent with

$$T(x,y) \le z \;\Leftrightarrow\; I_T(x,z) \ge y$$

for all $x, y, z \in [0,1]$. We will use these facts frequently in the sequel.

If T admits (T4) (e.g., if T is a t-norm) then N_T is a negation and called the *induced negation* of T. We say that a t-norm T has a strong induced negation if N_T is a strong negation. In Fig. 1 we present the three-dimensional plots of the product t-norm given by $T(x,y) = x \cdot y$ and an ordinal sum (see e.g. [8]) with one Lukasiewicz summand on $[0,0.4]$.

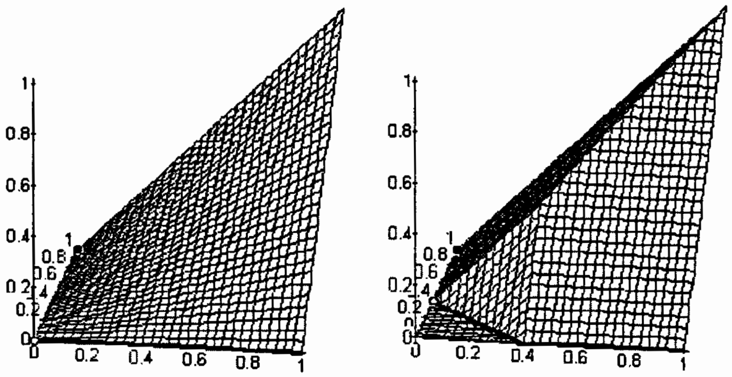

Figure 1: How the induced negation can be seen on the graph of the t-norm

One can see easily the induced negation of them on the plane which is spanned by the axes X and Y. It follows the border between the regions where T is zero and where T is positive. The border belongs to the zero region because of left continuity. None of the negations in the figures are strong.

If T is a left-continuous t-norm then the following (portation law) is fulfilled, for all $x, y, z \in [0,1]$:

$$I_T(x, I_T(y,z)) = I_T(T(x,y), z).$$

In addition, the residual implication 'defines the order', that is we have $I_T(x,y) = 1$ if and only if $x \leq y$.

3 The known examples of left-continuous t-norms with strong induced negations

Only three families of left-continuous t-norms with strong induced negations are known. (Under a family we understand a t-norm T together with its f-transformations which are as well t-norms and are defined by $T_f(x,y) = f^{-1}(T(f(x), f(y)))$, where f is any increasing bijection of $[0,1]$. That is, one family is a set of t-norms which are order-isomorphic from the semigroup theoretic sense.) One is the (continuous) nilpotent class, with as representative the Lukasiewicz t-norm given by

$$T(x,y) = \max(x+y-1, 0). \tag{2}$$

An other is the family of nilpotent minimum [3], with as representative

$$T(x,y) := \begin{cases} 0 & \text{if } y \leq 1-x \\ \min(x,y) & \text{otherwise} \end{cases}. \tag{3}$$

The third is the family of nilpotent ordinal sums [5]. This family (in wide sense) contains the two previous ones. A representative is given by

$$T(x,y)=\begin{cases} 0 & \text{if } x\le 1-y \\ \frac{1}{3}+x+y-1 & \text{if } \frac{1}{3}\le x,y\le \frac{2}{3} \text{ and } x>1-y \\ \min(x,y) & \text{otherwise} \end{cases}. \tag{4}$$

4 Rotation invariance and self quasi-inverse properties

First we introduce two properties.

Definition 1 Let $T\ :\ [0,1]^2\to[0,1]$ be a function satisfying (T1) and (T3), and let N be a strong negation. We say that T admits the rotation invariance property (with respect to N) if for all $x,y,z\in[0,1]$ we have

$$T(x,y)\le z\ \Leftrightarrow\ T(y,N(z))\le N(x).$$

Definition 2 Let $T\ :\ [0,1]^2\to[0,1]$ be a left-continuous function satisfying (T1) and (T3), and let N be a strong negation. We say that T admits the self quasi-inverse property (with respect to N) if for all $x,y,z\in[0,1]$ we have

$$I_T(x,y)=z\ \Leftrightarrow\ T(x,N(y))=N(z).$$

It is interesting to notice:

Theorem 1 *Any rotation invariant monotone binary operation is left-continuous. That is, any function $T\ :\ [0,1]^2\to[0,1]$ which satisfies (T3) and which is rotation invariant w.r.t. a strong negation N is left-continuous.*

Proof. First we prove left-continuity of T in its first component. By monotonicity of T the contrary is equivalent to the following statement: There exist $x,y,z\in[0,1]$ such that $\lim_{t\to z-}T(t,x)=y$ and $T(z,x)>y$. Hence by monotonicity, for all $t<z$ we have $T(t,x)\le y$. The rotation invariance of T implies now the following assertion: For all $t<z$ we have $T(x,N(y))\le N(t)$. This together with the involutory property of N yields $T(x,N(y))\le N(z)$ which is by applying rotation invariance of T twice is equivalent to $T(z,x)\le y$, a contradiction. The proof for the second component is analoguos. The only difference is that we have to apply the rotation invariance property first twice and then once (above it is applied first once then twice). ∎

The following proposition asserts that the only possible choice for N in the above two definitions is the induced negation N_T if T is a left-continuous t-norm. In the light of this proposition it is sufficient to say (without mentioning N) that a left-continuous t-norm T admits one of the properties defined above. Moreover, a left-continuous t-norm which admits one of these properties turns out to be a left-continuous t-norms with strong induced negation. Hence these properties are characteristic for the class of left-continuous t-norms with strong induced negations.

Proposition 1 *Let T be a left-continuous t-norm and N be a strong negation. If T admits either the rotation invariance property or the self quasi-inverse property with respect to N then $N = N_T$.*

Proof. If T admits the rotation invariance property with respect to N then let $z = 0$. We obtain $T(x,y) \leq 0 \Leftrightarrow T(y,1) = y \leq N(x)$. If T admits the self quasi-inverse property with respect to N then let $z = 1$. We obtain $x \leq y \Leftrightarrow I_T(x,y) = 1 \Leftrightarrow T(x, N(y)) = 0$. In both cases we conclude $N = N_T$ by (1). ∎

The importance of the above properties is due to the fact that, as we will see, any left-continuous t-norm with strong induced negation has the rotation invariance property and the self quasi-inverse property.

Theorem 2 *Let $T : [0,1]^2 \to [0,1]$ be a left-continuous function which satisfies (T1), (T2) and (T3) and suppose that N_T is a strong negation. Then for all $x, y, z \in [0,1]$ the following assertions hold:*

i. $I_T(x,y) = N_T(T(x, N_T(y)))$,

ii. $I_T(x,y) = z \Leftrightarrow T(x, N_T(y)) = N_T(z)$, (self quasi-inverse)

iii. $I_T(x,y) = I_T(N_T(y), N_T(x))$, (contrapositive symmetry of I_T)

iv. $T(x,y) \leq z \Leftrightarrow T(y, N_T(z)) \leq N_T(x)$, (rotation invariance)

In particular, if T is a left-continuous t-norm with strong induced negation N_T then i. – iv. hold.

Proof. Property i follows using the portation law and the involutive property of N_T:

$$N_T(T(x, N_T(y))) = I_T(T(x, I_T(y,0)), 0) = I_T(x, I_T(I_T(y,0),0)) = I_T(x,y).$$

The involutive property of N_T and i concludes ii. By i, commutativity of T and the involutive property of N_T we have

$$I_T(x,y) = N_T(T(x, N_T(y))) = N_T(T(N_T(y), x)) = N_T(T(N_T(y), N_T(N_T(x)))) = I_T(N_T(y), N_T(x))$$

which thus verifies iii. Finally, $T(x,y) \leq z$ is equivalent with $I_T(x,z) \geq y$ by left-continuity and it is equivalent with $I_T(N_T(z), N_T(x)) \geq y$ by iii. It holds if and only if $T(N_T(z), y) \leq N_T(x)$ by left-continuity again and that is finally, equivalent with $T(y, N_T(z)) \leq N_T(x)$ by commutativity. This ends the proof of iv. ∎

Corollary 1 *A left-continuous t-norm T has strong induced negation if and only if it admits one (and whence both) of the rotation invariance property and the self quasi-inverse property.*

Proof. See Proposition 1 and Theorem 2. ∎

5 Equivalence of the rotation invariance and the self quasi-inverse properties

Corollary 1 establishes the equivalence of the self quasi-inverse property and the rotation invariance property on the class of left-continuous t-norms. Indeed, the self quasi-inverse property is equivalent with strongness of the induced negation and it is equivalent with the rotation invariance property. Now we investigate the connection between the self quasi-inverse property and the rotation invariance property in a more general setting. These properties are equivalent in the following sense:

Theorem 3 *Let $T : [0,1]^2 \to [0,1]$ be a left-continuous function satisfying (T1) and (T3), and let N be a strong negation. Then the following assertions are equivalent. For all $x, y, z \in [0,1]$ we have*

i. $I_T(x,y) = z \Leftrightarrow T(x, N(y)) = N(z)$ *(self quasi-inverse property of T),*

ii. $T(x,y) \leq z \Leftrightarrow T(y, N(z)) \leq N(x)$ *(rotation invariance of T).*

Proof. a.) Suppose that T admits the rotation invariance property. Consider arbitrary $x, y, z \in [0,1]$.

First we prove that $T(x, N(y)) = N(z)$ implies $I_T(x,y) = z$. $T(x, N(y)) = N(z)$ implies $T(x, N(y)) \leq N(z)$ which, by applying the rotation invariance property twice and the symmetry of T, implies $T(x,z) \leq y$. By left-continuity we have $I_T(x,y) \geq z$. On the other hand $I_T(x,y) \geq I_T(x,y)$ and left-continuity implies $T(x, I_T(x,y)) \leq y$. By applying the rotation invariance property twice and the symmetry of T we get $T(x, N(y)) \leq N(I_T(x,y))$. That is, $N(z) \leq N(I_T(x,y))$. Since N is strong, this implies $I_T(x,y) \leq z$. Therefore, $I_T(x,y) = z$ as it is stated.

Now we prove that $I_T(x,y) = z$ implies $T(x, N(y)) = N(z)$. Indeed, we have $T(x, N(y)) = T(x, N(y))$ and by applying the above proved argument we obtain $I_T(x,y) = N(T(x, N(y)))$. That is, $z = N(T(x, N(y)))$ which, by applying that N is strong, implies $T(x, N(y)) = N(z)$, which is the other direction.

Summarizing, we have $T(x, N(y)) = N(z)$ if and only if $I_T(x,y) = z$ which shows that the self pseudo-inverse property holds.

b.) Suppose now that T admits the self pseudo-inverse property. Consider arbitrary $x, y, z \in [0,1]$.

We will prove that $T(x,y) \leq z$ implies $T(y, N(z)) \leq N(x)$. Indeed, $T(x,y) \leq z$ implies

$$N(T(x,y)) \geq N(z) \tag{5}$$

by the order reversing property of N. On the other hand, $T(y,x) = T(x,y)$ with the self quasi-inverse property yields $I_T(y, N(x)) = N(T(x,y))$. This ensures $I_T(y, N(x)) \geq N(T(x,y))$ which by left-continuity implies $T(y, N(T(x,y))) \leq N(x)$. The monotonicity of T and (5) ensures $T(y, N(z)) \leq N(x)$ as it is stated.

By applying the just proved argument twice with the involutive property of N we get that $T(y, N(z)) \leq N(x)$ ensures $T(x,y) \leq z$, the other direction.

Summarizing, we have $T(y, N(z)) \leq N(x) \Leftrightarrow T(x, y) \leq z$ which shows that the rotation invariance property holds. ∎

Two properties very similar to the ones defined here have already been investigated in [2]:

$$I_T(x, y) = N(T(x, N(y))) \tag{6}$$

instead of the self quasi-inverse property and

$$T(x, y) \leq z \Leftrightarrow T(x, N(z)) \leq N(y) \tag{7}$$

instead of the rotation invariance property. The right-hand side of (6) is well known in the literature and called the S-implication generated by T and N. Thus (6) expresses the equality of the S-implication and the residual implication (which is sometimes referred as R-implication). It is clear that (6) is equivalent with the self quasi-inverse property if and only if N is strong. In addition, (7) is equivalent with the rotation invariance property if and only if T is commutative. Hence, Theorem 3 can be deduced from Theorem 3.1 in [2]. Similarly, the if part of Corollary 1 can be deduced from Theorem 1 in [3] and Corollary 1 in [3]. *Equivalence* of i, iii in Theorem 2 (with a t-norm T and an arbitrary strong negation N) and (7) has already been shown in Theorem 1 in [3] but it was not recognised that properies $i - iv$ in Theorem 2 hold in fact when N is the induced negation N_T.

Our main aim in this paper is to present a geometric approach for the description of the class of left-continuous t-norms with strong induced negations, and in this framework the here defined properties are quite different from (6) and (7). As we will see, the rotation invariance property represents an order 3 transformation (a rotation), while — in the same sense — (7) represents an order 2 transformation (a reflection). The geometric meaning of the self quasi-inverse property (which will be described in the next section) can only be seen from its formulation in Definition 2, property (6) has a quite different geometric meaning. Namely, it says that the implication can be derived from the t-norm by two reflections of its graph, when the negation equals $1 - x$. (Each reflection belongs to a negation in formula (6)).

6 Geometrical interpretation of the rotation invariance and the self quasi-inverse property

The given names and the geometrical interpretation of the rotation invariance property and self quasi-inverse property are explained now.

Rotation invariance property. Let N be a strong negation. First observe that the transformation

$$\sigma : [0,1] \times [0,1] \times [0,1] \to [0,1] \times [0,1] \times [0,1] \text{ given by}$$

$$(x, y, z) \mapsto (y, N(z), N(x))$$

is of order 3. Therefore it can be considered as a rotation. In order to make this observation more visual, let $N(x) = 1 - x$ (the standard negation). Then an easy calculation verifies that σ is indeed a rotation of $[0,1]^3$ with angle $\frac{2\pi}{3}$ around the axis which is based on the points $(0,0,1)$ and $(1,1,0)$. By virtue of the above, formula $T(x,y) \leq z \Leftrightarrow T(y, N(z)) \leq N(x)$ means exactly that the part of the space $[0,1]^3$ which is *above the graph of* T remains invariant under σ. Or equivalently, we have $T(x,y) > z \Leftrightarrow T(y, N(z)) > N(x)$ for all $x, y, z \in [0,1]$, and it means exactly that the part of the space $[0,1]^3$ which is *strictly below the graph of* T remains invariant under σ.

In Fig. 2 the first row presents the three-dimensional plots of the t-norms given by (2), (3) and (4), respectively. Since their induced negations equal $1 - x$ one can recognize easily the geometrical meaning of the rotation invariance property.

Self quasi-inverse property. Now we recall the definition of quasi-inverses of decreasing real functions ([9]) in the form which is restricted to functions of type $[0,1] \to [0,1]$: Let $f : [0,1] \to [0,1]$ be a decreasing function. Let $f^* : [0,1] \to [0,1]$ be a function fulfilling the following conditions:

i. If y is in the range of f, then $f^*(y)$ is in $f^{-1}(\{y\})$.

ii. If y in not in the range of f, then (by declaring $\sup\{\emptyset\} = 0$)

$$f^*(y) = \sup\{t \in [0,1] \mid f(t) \geq y\}.$$

Call f^* a quasi-inverse of f. Generally, such an f^* is not unique. Clearly, if f is a bijection of $[0,1]$ then f^* is unique and coincides with f^{-1}, the usual inverse of f.

We are going to prove that for any left-continuous t-norm T the negation of any partial mapping $T(x,\cdot)$ is the quasi-inverse of itself if and only if T admits the self quasi-inverse property. This fact motivates the name of this property in Definition 2.

Proposition 2 *Let T be left-continuous t-norm. Then the (induced) negation of the x-partial mapping defined by*

$$f_x : [0,1] \to [0,1], \quad y \mapsto N_T(T(x,y))$$

is the quasi-inverse of itself for all $x \in [0,1]$ if and only if T admits the self quasi-inverse property.

Proof. Note that by Proposition 1 and Theorem 2, if T admits the self quasi-inverse property with respect to N then $N = N_T$ and T is a left-continuous t-norm with strong induced negation N_T. Clearly, f_x is decreasing. If y is in the range of f_x then we need to verify $f_x(f_x(y)) = y$. Since y is in the range of f_x, there exists $z \in [0,1]$ such that $y = N_T(T(x,z))$. Hence, we have $N_T(y) = T(x,z)$ by the involutive property of N_T. $N_T(T(x,y)) = I_T(x, N_T(y))$ for all $x, y \in [0,1]$ if and only if the self quasi-inverse property holds. The definition of residuation ensures:

$T(t, I_T(t,u)) = u$ for all $t, u \in [0,1]$ if and only if u is in the range of the partial mapping $T(t,\cdot)$. The above arguments with the definition of f_x yield that we have

$$\begin{aligned} f_x(f_x(y)) &= N_T(T(x, N_T(T(x,y)))) \\ &= N_T(T(x, I_T(x, N_T(y)))) = N_T(N_T(y)) = y \end{aligned}$$

if and only if the self quasi-inverse property holds.

If y is not in the range of f_x then we have

$$\begin{aligned} f_x(y) = N_T(T(x,y)) = I_T(x, N_T(y)) &= \sup\{t \in [0,1] \mid T(x,t) \le N_T(y)\} \\ = \sup\{t \in [0,1] \mid N_T(T(x,t)) \ge y\} &= \sup\{t \in [0,1] \mid f_x(t) \ge y\} \end{aligned}$$

again if and only if the self quasi-inverse property holds. This ends the proof. ■

In the frame of real functions the quasi-inverse has a geometrical interpretation. There is a simple geometrical way to construct the graph of the quasi-inverse f^* form the graph of f.

i. Draw vertical line segments at the discontinuities of f.

ii. Reflect the obtained graph at the first median, i.e., the graph of the identity function.

iii. Remove all vertical line segments from the reflected graph except for one point (this has to be done in such a way that $f^*(y) \in f^{-1}(\{y\})$) is satisfied).

Now, let $N(x) = 1 - x$. Then the geometrical interpretation of the negation is the reflection of the graph at the line given by $y = \frac{1}{2}$. In this case, the graph of any partial mapping $T(x,\cdot)$ has the following geometrical property. First extend the discontinuities of $T(x,\cdot)$ with vertical line segments. Then the obtained graph is invariant under the reflection at the second median (given by $y = 1 - x$).

In Fig. 2 the second row presents plots of the partial mappings $T(x, \frac{1}{2})$ of the t-norms given by (2), (3) and (4), respectively. Again, since their induced negations equal $1 - x$ one can recognise easily the geometrical meaning of the self quasi-inverse property.

Remark 1 For any t-norm T and strong negation N we can define the N-dual of T by $S(x,y) = N(T(N(x), N(y)))$. Then S is a t-conorm and clearly, left-continuity of T is equivalent with right-continuity of S. We remark that all the results proved in this paper have their equivalent counterparts for t-conorms.

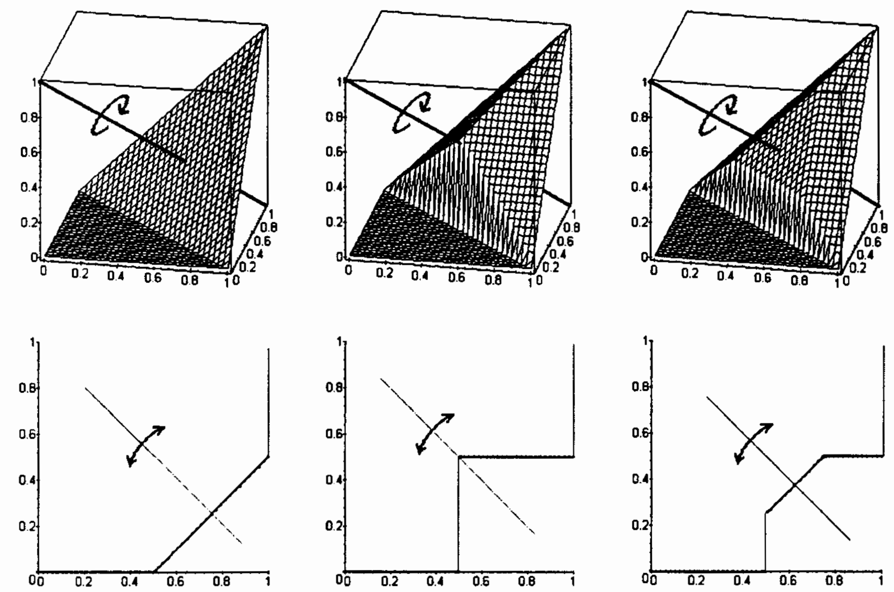

Figure 2: Geometrical interpretation of the rotation invariance property and the self quasi-inverse property

7 Conclusion

In this paper we have proved that a left-continuous t-norm has a strong induced negation if and only if it admits the self quasi-inverse property, or equivalently, if and only if it admits the rotation invariance property. Further, fulfilling the self quasi-inverse property w.r.t. $1-x$ means exactly that the graph of any partial mapping of the t-norm has the following geometrical property: its graph is invariant under the reflection at the second median (given by $y = 1-x$) if we first extend the discontinuities of the partial mapping with vertical line segments. In addition, fulfilling the rotation invariance property of a t-norm means exactly that the part of the space $[0,1] \times [0,1] \times [0,1]$ which is strictly below the graph of it remains invariant under an order 3 transformation. This transformation is a rotation of $[0,1]^3$ (with angle $\frac{2\pi}{3}$ around the axis which is based on the points $(0,0,1)$ and $(1,1,0)$) when the induced negation of the t-norm is $1-x$. It turned out that the two properties are equivalent on the class of symmetric, non-decreasing two-place functions on $[0,1]$, that is, such a function admits the rotation invariance property if and only if it admits the self quasi-inverse property.

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