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Optimization under Fuzzy Rule Constraints

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Abstract

Suppose we are given a mathematical programming problem in which the functional relationship between the decision variables and the objective function is not completely known. Our knowledge-base consists of a block of fuzzy if-then rules, where the antecedent part of the rules contains some linguistic values of the decision variables, and the consequence part is a linear combination of the crisp values of the decision variables. We suggest the use of Takagi and Sugeno fuzzy reasoning method to determine the crisp functional relationship between the objective function and the decision variables, and solve the resulting (usually nonlinear) programming problem to find a fair optimal solution to the original fuzzy problem.

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1 Introduction

Fuzzy optimization problems can be stated and solved in many different ways [2, 3, 14, 21, 22]. Usually the authors consider optimization problems of the form

$$\max/\min\ f(x);\ \text{subject to } x \in X,$$

where f or/and X are defind by fuzzy terms. Then they are searching for a crisp x^* which (in a certain) sense maximizes f on X. For example, fuzzy linear programming (FLP) problems can be stated as [9, 13, 15, 17]

$$\max/\min\ f(x) = \tilde{c}x;\ \text{subject to } \tilde{A}x \lesssim \tilde{b}, \tag{1}$$

where the fuzzy terms are denoted by tilde. Fullér and Zimmermann [12] interpreted FLP problems (1) with fuzzy coefficients and fuzzy inequality relations as multiple fuzzy reasoning schemes, where the antecedents of the scheme correspond to the constraints of the FLP problem and the fact of the scheme is the objective of the FLP problem. Their solution process consists of two steps: first, for every decision variable $x \in \mathbb{R}^n$, compute the (fuzzy) value of the objective function, $\mathrm{MAX}(x)$, via sup-min convolution of the antecedents/constraints and the fact/objective. Then an (optimal) solution to the FLP problem is any point which produces a maximal element of the set $\{\mathrm{MAX}(x) \mid x \in \mathbb{R}^n\}$.

Unlike in (1) the fuzzy value of the objective function $f(x)$ may not be known for any $x \in \mathbb{R}^n$. More often than not we are only able to describe the causal link between x and $f(x)$ linguistically using fuzzy if-then rules.

In [8] we have considered constrained fuzzy optimization problems of the form

$$\max/\min\ f(x);\ \text{subject to } \{\Re_1(x), \dots, \Re_m(x) \mid x \in X \subset \mathbb{R}^n\}, \tag{2}$$

with

$$\Re_i(x) : \text{if } x_1 \text{ is } A_{i1} \text{ and } \dots \text{ and } x_n \text{ is } A_{in} \text{ then } f(x) \text{ is } C_i,$$

where A_{ij} and C_i are fuzzy numbers; and we have suggested the use of Tsukamoto's fuzzy reasoning method [19] to determine the crisp values of f.

In this paper we suppose that our knowledge base contains fuzzy if-then rules of the form

$$\Re_i(x) : \text{if } x_1 \text{ is } A_{i1} \text{ and } \dots \text{ and } x_n \text{ is } A_{in} \text{ then } f(x) = a_{i1}x_1 + \dots + a_{in}x_n + b_i \tag{3}$$

where A_{ij} is a fuzzy number, and a_{ij} and b_i are real numbers. Then we determine the crisp value of f at $u \in \mathbb{R}^n$ by the Takagi and Sugeno fuzzy reasoning method, and obtain an optimal solution to (2) by solving the resulting (usually nonlinear) optimization problem

$$\max/\min\ f(u),\ \text{subject to } u \in X.$$

We illustrate the proposed method by several examples.

2 Constrained Optimization under Fuzzy If-then Rules

A linguistic variable [20] can be regarded either as a variable whose value is a fuzzy number or as a variable whose values are defined in linguistic terms. A fuzzy set A in X is called a fuzzy point if there exists a $u \in X$ such that $A(t) = 1$ if $t = u$ and $A(t) = 0$ otherwise. We will use the notation $A = \bar{u}$. Fuzzy points are used to represent crisp values of linguistic variables. If x is a linguistic variable in the universe of discourse X and $u \in X$ then we simple write "$x = u$" or "x is $\bar{u}$" to indicate that u is a crisp value of x.

To find a fair solution to the fuzzy optimization problem

$$\max/\min\ f(x);\ \text{subject to } \{\Re_1(x), \dots, \Re_m(x) \mid x \in X\}, \tag{4}$$

with fuzzy if-then rules of form (3) we first determine the crisp value of the objective function f at $u \in \mathbb{R}^n$, denoted also by $f(u)$, by the compositional rule of inference

$$f(u) := (x \text{ is } \bar{u}) \circ \{\Re_1(x), \cdots, \Re_m(x)\}$$

using the Takagi and Sugeno fuzzy reasoning method as

$$f(u) := \frac{\alpha_1 z_1(u) + \cdots + \alpha_m z_m(u)}{\alpha_1 + \cdots + \alpha_m}.$$

where the firing levels of the rules are computed by

$$\alpha_i = \prod_{j=1}^{n} A_{ij}(u_j), \tag{5}$$

and the individual rule outputs, denoted by z_i, are derived from the relationships

$$z_i(u) = \sum_{j=1}^{n} a_{ij} u_j + b_i.$$

To determine the firing level of the rules, we suggest the use of the product t-norm (to have a smooth output function). In this manner our constrained optimization problem (4) turns into the following crisp (usually nonlinear) mathematical programmimg problem

$$\max/\min\ f(u);\ \text{subject to } u \in X.$$

If X is a fuzzy set with membership function μ_X (e.g. given by soft constraints as in [21]) then following Bellman and Zadeh [1] we define the fuzzy solution to problem (4) as

$$D = \mu_X \cap \mu_f, \tag{6}$$

where μ_f is an appropriate transformation of the values (computed by the Takagi and Sugeno reasoning method) of f to the unit interval [10], and an optimal solution to (4) is defined to be as any maximizing element of D.

Example 1. Consider the optimization problem

$$\max f(x);\ \text{subject to}\ \{\Re_1(x), \Re_2(x) \mid x \in X = [0,1]\}, \tag{7}$$

where

$$\begin{aligned} \Re_1(x):&\ \text{if } x \text{ is } \textit{small} \text{ then } && f(x) = x \\ \Re_2(x):&\ \text{if } x \text{ is } \textit{big} \text{ then } && f(x) = 1 - x. \end{aligned}$$

If $\text{small}(x) = 1 - x$ and $\text{big}(x) = x$, and u is an input to the rule base then the firing levels of the rules are computed by

$$\alpha_1 = 1 - u, \alpha_2 = u.$$

Then we get

$$f(u) = (1-u)u + u(1-u) = 2(1-u)u.$$

Thus our original fuzzy problem turns into the following crisp nonlinear mathematical programming problem

$$\max\ 2u(1-u);\ \text{subject to}\ u \in [0,1].$$

which has the optimal solution $u^* = 1/2$.

If the membership functions in the rules are defined by

$$\text{small}(x) = \frac{1}{1 + e^{-(1/2-x)}}, \quad \text{big}(x) = \frac{1}{1 + e^{(1/2-x)}}$$

and u is an input to the rule base then the firing levels of the rules are computed by

$$\alpha_1 = \frac{1}{1 + e^{-(1/2-u)}}, \quad \alpha_2 = \frac{1}{1 + e^{(1/2-u)}}.$$

Then we get

$$f(u) = \frac{u}{1 + e^{-(1/2-u)}} + \frac{1-u}{1 + e^{(1/2-u)}}.$$

Thus our original fuzzy problem turns into the following crisp nonlinear mathematical programming problem

$$\max\left(\frac{u}{1 + e^{-(1/2-u)}} + \frac{1-u}{1 + e^{(1/2-u)}}\right);\ \text{subject to}\ u \in [0,1].$$

which has the optimal solution $u^* = 1/2$.

If we use nonsymmetric membership functions in the rules, for example

$$\text{small}(x) = \frac{1}{1 + e^{-10(3/4-x)}}, \quad \text{big}(x) = \frac{1}{1 + e^{0.1(1/3-x)}}$$

and u is an input to the rule base then the firing levels of the rules are computed by

$$\alpha_1 = \frac{1}{1+e^{-10(3/4-u)}}, \quad \alpha_2 = \frac{1}{1+e^{0.1(1/3-u)}}.$$

Then we get

$$f(u) = \frac{\dfrac{u}{1+e^{-10(3/4-u)}} + \dfrac{1-u}{1+e^{0.1(1/3-u)}}}{\dfrac{1}{1+e^{-10(3/4-u)}} + \dfrac{1}{1+e^{0.1(1/3-u)}}}.$$

Thus our original fuzzy problem turns into the following crisp nonlinear mathematical programming problem

$$\max \left(\frac{\dfrac{u}{1+e^{-10(3/4-u)}} + \dfrac{1-u}{1+e^{0.1(1/3-u)}}}{\dfrac{1}{1+e^{-10(3/4-u)}} + \dfrac{1}{1+e^{0.1(1/3-u)}}} \right); \text{ subject to } u \in [0,1].$$

which has the optimal solution $u^* = 0.65$ and $f(u^*) = 0.52$..

Even though Example 1 is probably the simplest one, it clearly shows the complexity of the problem of optimization under fuzzy if-then rules. Namely, the only way to increase $f(u)$ is to decrease the feasibility of u.

Example 2. Consider the optimization problem

$$\min f(x); \text{ subject to } \{x_1 + x_2 = 1/2,\ 0 \le x_1, x_2 \le 1\}, \tag{8}$$

where

$$\begin{array}{ll} \Re_1(x): & \text{if } x_1 \text{ is } \textit{small} \text{ and } x_2 \text{ is } \textit{small} \text{ then } \quad f(x) = x_1 + x_2, \\ \Re_2(x): & \text{if } x_1 \text{ is } \textit{small} \text{ and } x_2 \text{ is } \textit{big} \text{ then } \quad f(x) = -x_1 + x_2. \end{array}$$

Let $u = (u_1, u_2)$ be an input to the fuzzy system. Then the firing levels of the rules are

$$\alpha_1 = (1-u_1)(1-u_2), \quad \alpha_2 = (1-u_1)u_2,$$

It is clear that if $u_1 = 1$ then no rule applies because $\alpha_1 = \alpha_2 = 0$. So we can exclude the value $u_1 = 1$ from the set of feasible solutions. The individual rule outputs are computed by

$$z_1 = u_1 + u_2, \quad z_2 = -u_1 + u_2.$$

and, therefore, the overall system output, interpreted as the crisp value of f at u is

$$f(u) = \frac{(1-u_1)(1-u_2)(u_1+u_2) + (1-u_1)u_2(-u_1+u_2)}{(1-u_1)(1-u_2) + (1-u_1)u_2} =$$

$$u_1 + u_2 - 2u_1u_2.$$

Thus our original fuzzy problem turns into the following crisp nonlinear mathematical programming problem

$$\min\ (u_1 + u_2 - 2u_1u_2);\ \text{subject to}\ \{u_1 + u_2 = 1/2,\ 0 \le u_1 < 1,\ 0 \le u_2 \le 1\}.$$

which has the optimal solution $u_1^* = u_2^* = 1/4$ and its optimal value is $f(u^*) = 3/8$.

Even though the individual rule outputs are linear functions of u_1 and u_2, the computed input/output function $f(u) = u_1 + u_2 - 2u_1u_2$ is a nonlinear one.

Example 3. Consider the problem

$$\max_X f \tag{9}$$

where X is a fuzzy susbset of the unit interval with membership function

$$\mu_X(u) = 1 - (1/2 - u)^2$$

for $u \in [0,1]$ and the fuzzy rules are

$$\begin{aligned} \Re_1(x):&\ \text{if } x \text{ is } \textit{small} \text{ then } & f(x) = 1 - x, \\ \Re_2(x):&\ \text{if } x \text{ is } \textit{big} \text{ then } & f(x) = x. \end{aligned}$$

Let $u \in [0,1]$ be an input to the fuzzy system $\{\Re_1(x), \Re_2(x)\}$. Then the firing levels of the rules are $\alpha_1 = 1-u, \alpha_2 = u$. The individual rule outputs are $z_1 = (1-u)(1-u)$, $z_2 = u^2$ and, therefore, the overall system output is

$$f(u) = (1-u)^2 + u^2 = 2u^2 + 2u + 1.$$

Then according to (6) our original fuzzy problem (9) turns into the following crisp biobjective mathematical programming problem

$$\max \min\{2u^2 + 2u + 1, 1 - (1/2 - u)^2\};\ \text{subject to}\ u \in [0,1],$$

which has the optimal value of 0.8333 and two optimal solutions $\{0.09, 0.91\}$.

The rules represent our knowledge-base for the fuzzy optimization problem. The fuzzy partitions for lingusitic variables will not ususally satisfy ε-completeness, normality and convexity. In many cases we have only a few (and contradictory) rules. Therefore, we can not make any preselection procedure to remove the rules which *do not play any role* in the optimization problem. All rules should be considered when we derive the crisp values of the objective function. We have chosen the Takagi and Sugeno fuzzy reasoning scheme, because the individual rule outputs are crisp functions, and therefore, the functional relationship between the input vector u and the system output $f(u)$ can be easily identified.

3 Summary

We have addressed mathematical programming problems in which the functional relationship between the decision variables and the objective function is known linguistically. We suggested the use of the Takagi and Sugeno fuzzy reasoning method to determine the crisp functional relationship between the objective function and the decision variables, and solve the resulting (usually nonlinear) programming problem to find a fair optimal solution to the original fuzzy problem. We can refine the fuzzy rule base by introducing new lingusitic variables modeling the linguistic dependencies between the variables and the objectives [4, 5, 6, 11]. These will be the subjects of our future research.

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