The Logical Representation of the Discrete Choquet Integral

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Abstract

A discrete Choquet integral can be used for the aggregation of attribute values in decision problems in the presence of interaction among attributes. The main drawback is the non-existence of a clear interpretation of what a given fuzzy measure means in terms of behavior in decision making. This paper presents an equivalent representation: the logical representation of the discrete Choquet integral. This logical representation enables expressing what a given fuzzy measure means in terms of behavior in decision making and/or enables a consistent explanation of a decision maker's preference structure in the presence of interaction among attributes.

Keywords: Fuzzy measure, Choquet integral, Multi-attribute decision making, Aggregation,Logical representation.

1 Introduction

Aggregation of attribute (feature, symptom, aspect, ...) values of an object (alternative, action, class, ...) is an important problem in many disciplines: multi-attributes decision making (MADM), pattern recognition, multivariable statistical analysis (MVSA), etc. The presence of interactions between attributes is a conceptual problem for classical aggregation techniques.

Fuzzy measures and fuzzy integrals offer a great potential as tools for aggregating attribute values in the presence of interactions between attributes. A discrete Choquet integral is used for pattern classification, feature extraction [5], in multicriteria decision making [6]. In [7] the author states that:

> Although fuzzy measures constitute a flexible tool for modelling the importance of coalitions (interactions among attributes), they are not easy to handle in a practical problem for the following two reasons:
>
> • "....too complex to handle if n goes beyond, say 8...
>
> •....if a fuzzy measure is given, nobody can tell exactly what it means in terms of behavior in decision making... ".

In this paper an equivalent representation of the discrete Choquet integral, *the logical representation of the discrete Choquet integral*, is proposed as the solution to this problem.

The logical representation of the discrete Choquet integral is based on the following three properties:

(a) linearity of the discrete Choquet integral by measures [13, 3];

(b) The Choquet integral for a logical (0,1) fuzzy measure is equivalent to the logical expression of attributes [12]; and

(c) any fuzzy measure (vector) can be represented as a convex combination of logical fuzzy measures (vectors).

The proposed equivalent representation consists of a linear convex combination of continuous logical expressions of attributes. The logical expression over the relevant elements of the power set of attributes contains AND and/or OR operators, with AND defined as min and OR as max.

The equivalent logical representation is much more convenient than a classical discrete Choquet integral for consistently expressing the preference structure (or structure of wishes) of a decision maker in MADM. Instead of dealing with a fuzzy measure, without a clear interpretation in general, by using the equivalent logical representation, the problem is reduced to the extension of a list of attributes by appropriate logical expressions on attributes. In a simple example is illustrated the advantage of the equivalent logical representation of discrete Choquet integral for resolving MADM problems.

The well known definitions of a discrete fuzzy measure and a discrete Choquet integral are given in Section 2. The interpretation and understanding of the discrete Choquet integral based on the logical representation are given in Section 3. The application of the logical representation of the discrete Choquet integral to multi-attribute decision making is illustrated on two examples in Section 4. In Section 5, a logical Möbius transform and a logical interaction index for a logical measure are introduced.

2 Fuzzy Measures and the Choquet Integral

In this paper only discrete spaces are considered, and the finite universe Ω of n elements (attributes, features,...), $\Omega = \{a_1, ..., a_n\}$. $\mathcal{P}(\Omega)$ is the power set of Ω, while $|A|$ denotes the cardinality of a subset A of Ω, and $A \setminus B$ denotes the set difference. $\wedge$, $\vee$ denote min and max, respectively.

The additivity property for (probability) measures is usually a hard constraint for real problems. Sugeno [16] introduced fuzzy measures and integrals, as a generalization of the usual definition of a measure by relaxing the additivity property. The concept of a fuzzy measure is closely related to the twenty years older concept of a capacity, proposed by Choquet [1]. Fuzzy measures include as particular cases probability measures, possibility and necessity measures, belief and plausibility functions, etc. [5].

Definition 1 *A fuzzy measure μ on Ω is a mapping $\mu : \mathcal{P}(\Omega) \to [0,1]$ such that, for every A and B in $\mathcal{P}(\Omega)$:*

1. $\mu(\emptyset) = 0$,
2. *if $B \subseteq A$, then $\mu(B) \leq \mu(A)$,*

where: Ω is any set of elements, $\mathcal{P}(\Omega)$ is the set of fuzzy subsets of Ω, and $A, B, \ldots$ are subsets of Ω.

Definition 2 *The discrete Choquet integral of $(a_1, ..., a_n), a_i \in R$, with respect to μ is defined by*

$$C\mu(a_1, ..., a_n) := \sum_{i=1}^{n} \left(a_{(i)} - a_{(i-1)}\right) \mu\left(A_{(i)}\right)$$

where $\cdot_{(i)}$ indicates that the indices have been permuted so that $0 \leq a_{(1)} \leq \ldots \leq a_{(n)}$, and $A_{(i)} := \left\{a_{(i)}, ..., a_{(n)}\right\}$, and $a_{(0)} = 0$.

The Choquet integral is a generalization of the Lebesgue integral, and it coincides with the Lebesgue integral when the measure is additive. A discrete Choquet integral enables modeling of positive interaction and redundancy between attributes.

In [7], the author states that: "if a fuzzy measure is given, nobody can tell exactly what it means in terms of behavior in decision making".

In the next section a meaning of a fuzzy measure related to discrete Choquet integral is proposed by its logical representation.

3 Logical Interpretation of the Discrete Choquet Integral

An equivalent representation, the logical representation of the discrete Choquet integral, is presented in this section.

The logical representation of the discrete Choquet integral is based on the following three properties:

(a) linearity of the discrete Choquet integral by measures [13, 3]; (If a fuzzy measure can be represented as a linear convex combination of some other fuzzy measures then the Choquet integral for this measure is equivalent to the linear convex combination of the Choquet integrals for those fuzzy measures)

(b) the Choquet integral for a logical (0,1) fuzzy measure is equivalent to a logical expression of attributes [12], and

(c) any fuzzy measure (vector) can be represented as a convex combination of logical fuzzy measures (vectors).

3.1 Linearity of the discrete Choquet integral by measures

Linearity of the discrete Choquet integral by measures, can be formalized as follows:

Proposition 1 *Let μ, $\mu_1, ..., \mu_Q$ be fuzzy measures and $\mu = \sum_{q=1}^{Q} \lambda_q \mu_q$, where $\sum_{q=1}^{Q} \lambda_q = 1$ and $\lambda_q \geq 0, q = 1, ..., Q$. Then for the discrete Choquet integrals of $(a_1, ..., a_n)$ w.r.t. μ, $\mu_1, ..., \mu_Q$ the following holds:*

$$C\mu\left(a_1, ..., a_n\right) = \sum_{q=1}^{Q} \lambda_q C_{\mu_q}\left(a_1, ..., a_n\right).$$

3.2 A logical fuzzy measure

Logical fuzzy measures are very important for the explanation of the discrete Choquet integral, via an equivalent representation - the logical representation of the Choquet integral. A logical fuzzy measure is defined as:

Definition 3 *A logical fuzzy measure is fuzzy measure that takes its values in* $\{0, 1\}$.

3.3 A logical Choquet integral

A discrete Choquet integral for a logical fuzzy measure is a logical expression on the attributes with *and* and *or* operators, defined as min and max respectively.

Definition 4 *A logical Choquet integral is the Choquet integral for a logical fuzzy measure.*

A logical Choquet integral takes the following form [12]:

$$C_{\mu^L}\left(a_1, ..., a_n\right) = \bigvee_{A:\mu^L(A)=1} \left(\bigwedge_{a_k \in A} a_k\right)$$

Example 1 *A. The Choquet integral for the logical fuzzy measure* $\mu_1 = 1,\ \mu_2 = 0$ *and* $\mu_{12} = 1$, *of two attributes, is*

$$\begin{aligned} C\mu(a_1, a_2) &= a_1 \vee (a_1 \wedge a_2) \\ &= a_1 \end{aligned}$$

B. The Choquet integral for the logical fuzzy measure $\mu_1 = 1,\ \mu_2 = 0,\ \mu_3 = 1,$ $\mu_{12} = 1,\ \mu_{13} = 1,\ \mu_{23} = 1$ *and* $\mu_{123} = 1$, *of three attributes, is*

$$\begin{aligned} C\mu(a_1, a_2, a_3) &= a_1 \vee a_3 \vee (a_1 \wedge a_2) \vee (a_1 \wedge a_3) \vee (a_2 \wedge a_3) \vee (a_1 \wedge a_2 \wedge a_3) \\ &= a_1 \vee a_3. \end{aligned}$$

Modeling by the Choquet integral the AND, OR and ONLY functions for two attributes is useful for an illustrative explanation of the logical representation of the Choquet integral.

3.3.1 AND and OR modeling by Choquet integral

The "extreme" cases of positive interaction and redundancy between two attributes are AND and OR functions, respectively.

The values of fuzzy measures, for the case when AND and OR operators are modeled using the Choquet integral, are given through the following examples.

Example 2 *Find a discrete fuzzy measure for the aggregation of the values of attributes* a_1 *and* a_2, *using a discrete Choquet integral, as an AND operator. Necessary conditions for the Choquet integral to model an AND operator, in the case of two attributes are:*

a_1	a_2	$C\mu(a_1, a_2)$
1	*1*	*1*
1	*0*	*0*
0	*1*	*0*
0	*0*	*0*

From the necessary conditions for an AND function and from the definition of the discrete Choquet integral, a logical fuzzy measure is defined as:

$$\begin{array}{ll} C\mu(1,1) = \mu_{12} & \Rightarrow \mu_{12} = 1 \\ C\mu(0,1) = \mu_2 & \Rightarrow \mu_2 = 0 \\ C\mu(1,0) = \mu_1 & \Rightarrow \mu_1 = 0 \end{array} .$$

The Choquet integral for arbitrary values of attributes and for a fuzzy measure of AND function, is:

$$\begin{aligned} C\mu(a_1, a_2) &= a_{(1)} \\ &= \min(a_1, a_2) \\ &= a_1 \wedge a_2. \end{aligned}$$

Example 3 *Find a discrete fuzzy measure for the aggregation of the values of attributes a_1 and a_2, using a discrete Choquet integral, as an OR operator. Necessary conditions for the Choquet integral to model an OR operator, in the case of two attributes are:*

a_1	a_2	$C\mu(a_1, a_2)$
1	1	1
1	0	1
0	1	1
0	0	0

From the necessary conditions for an OR relation and from the definition of the discrete Choquet integral, a logical fuzzy measure is defined as:

$$\begin{array}{ll} C\mu(1,1) = \mu_{12} & \Rightarrow \mu_{12} = 1 \\ C\mu(0,1) = \mu_2 & \Rightarrow \mu_2 = 1 \\ C\mu(1,0) = \mu_1 & \Rightarrow \mu_1 = 1 \end{array}.$$

The Choquet integral for arbitrary values of attributes and for a fuzzy measure of OR operator is:

$$\begin{aligned} C\mu(a_1, a_2) &= a_{(2)} \\ &= \max(a_1, a_2) \\ &= a_1 \vee a_2. \end{aligned}$$

3.3.2 $ONLY_{a_1}$ and $ONLY_{a_2}$ modeling by Choquet integral

Fuzzy measures for modeling functions

$$ONLY_{a_i}(a_1, a_2) := a_i; \quad i = 1, 2$$

by aggregation with a fuzzy Choquet integral are given through the following example.

Example 4 *Find a discrete fuzzy measure for the aggregation of the values of attributes a_1 and a_2, using a discrete Choquet integral, as an $ONLY_{a_1}$ ($ONLY_{a_2}$) operator. A discrete Choquet integral models an $ONLY_{a_1}$ operator in case of two attributes, if:*

$$C\mu(a_1, a_2) = a_1, \ \forall a_1, a_2$$

From the condition for an $ONLY_{a_1}$ operator and from the definition of the discrete Choquet integral, a logical fuzzy measure is defined as:

$$\begin{array}{ll} a_1\mu_{12} + (a_2 - a_1)\mu_2 = a_1, \ \textit{for } a_1 \le a_2 & \Rightarrow \mu_2 = 0 \\ a_2\mu_{12} + (a_1 - a_2)\mu_1 = a_1, \ \textit{for } a_2 < a_1 & \Rightarrow \mu_1 = 1 \end{array}.$$

In the same way it could be shown that to an operator $ONLY_{a_2}$ corresponds a logical fuzzy measure $\mu_1 = 0$, $\mu_2 = 1$, $\mu_{12} = 1$.

3.4 Fuzzy measure vector and fuzzy measure space

Any fuzzy measure on a finite set of attributes Ω can be represented as a vector $\overrightarrow{\mu}$, with 2^n components, where $n = |\Omega|$.

Definition 5 *Fuzzy measure vector components, of a finite set of attributes Ω, are measures of the elements of the power set $\mathcal{P}(\Omega)$.*

The first and the last fuzzy measure vector components are fixed values, $\mu(\emptyset) = 0$ and $\mu(\Omega) = 1$, respectively.

Example 5 *A fuzzy measure vector for a two-attribute case is*

$$\begin{aligned}\overrightarrow{\mu} &= [\mu(\emptyset) = 0 \quad \mu(\{a_1\}) \quad \mu(\{a_2\}) \quad \mu(\{a_1, a_2\}) = 1]^T \\ &= [0 \quad \mu_1 \quad \mu_2 \quad 1]^T .\end{aligned}$$

Definition 6 *A logical fuzzy measure vector $\overrightarrow{\mu^L}$, is a fuzzy measure vector whose components take only values from $\{0, 1\}$.*

Example 6 *From the previous examples for the two-attribute case, logical fuzzy measure vectors and logical functions corresponding to the Choquet integrals are given in the following table:*

	μ_1	μ_2	μ_{12}	$C_{\overrightarrow{\mu^L}}(a_1, a_2)$
$\overrightarrow{\mu^L}_{a_1 \wedge a_2}$	*0*	*0*	*1*	$a_1 \wedge a_2$
$\overrightarrow{\mu^L}_{a_1}$	*1*	*0*	*1*	a_1
$\overrightarrow{\mu^L}_{a_2}$	*0*	*1*	*1*	a_2
$\overrightarrow{\mu^L}_{a_1 \vee a_2}$	*1*	*1*	*1*	$a_1 \vee a_2$

Definition 7 *A space defined by all possible fuzzy measure vectors is a fuzzy measure space.*

Example 7 *A fuzzy measure subspace, in the case of two attributes, of interest for the analysis is two dimensional, (μ_1, μ_2), fig.1, since it is by definition $\mu_0 = \mu(\emptyset) = 0$, and $\mu_{12} = \mu(\{a_1, a_2\}) = 1$.*

From previous examples: Points A and C, in fig.1 are logical fuzzy measure vectors for AND and OR functions, respectively, when the Choquet integral is used for aggregation of the values of two attributes. Points B and D in fig.1 represent the logical fuzzy measure vectors of the $ONLY_{a_1}$ and $ONLY_{a_2}$ operators, respectively, when the Choquet integral is used for the aggregation of the values of two attributes.

Example 8 *Logical fuzzy measure vectors for the three-attribute case and logical functions corresponding to the Choquet integrals are given in the following table:*

	μ_1	μ_2	μ_3	μ_{12}	μ_{13}	μ_{23}	μ_{123}	$C_{\overrightarrow{\mu^L}}(a_1, a_2, a_3)$
$\overrightarrow{\mu^L}_{a_1 \wedge a_2 \wedge a_3}$	0	0	0	0	0	0	1	$(a_1 \wedge a_2 \wedge a_3)$
$\overrightarrow{\mu^L}_{a_1 \wedge a_2}$	0	0	0	1	0	0	1	$(a_1 \wedge a_2)$
$\overrightarrow{\mu^L}_{a_1 \wedge a_3}$	0	0	0	0	1	0	1	$(a_1 \wedge a_3)$
$\overrightarrow{\mu^L}_{a_2 \wedge a_3}$	0	0	0	0	0	1	1	$(a_2 \wedge a_3)$
$\overrightarrow{\mu^L}_{a_1 \wedge (a_2 \vee a_3)}$	0	0	0	1	1	0	1	$a_1 \wedge (a_2 \vee a_3)$
$\overrightarrow{\mu^L}_{a_2 \wedge (a_1 \vee a_3)}$	0	0	0	1	0	1	1	$a_2 \wedge (a_1 \vee a_3)$
$\overrightarrow{\mu^L}_{a_3 \wedge (a_1 \vee a_2)}$	0	0	0	0	1	1	1	$a_3 \wedge (a_1 \vee a_2)$
$\overrightarrow{\mu^L}_{(a_1 \wedge a_2) \vee (a_1 \wedge a_3) \vee (a_2 \wedge a_3)}$	0	0	0	1	1	1	1	$\bigvee_{i,j=1,2,3}(a_i \wedge a_j)$
$\overrightarrow{\mu^L}_{a_1}$	1	0	0	1	1	0	1	a_1
$\overrightarrow{\mu^L}_{a_2}$	0	1	0	1	0	1	1	a_2
$\overrightarrow{\mu^L}_{a_3}$	0	0	1	0	1	1	1	a_3
$\overrightarrow{\mu^L}_{a_1 \vee (a_2 \wedge a_3)}$	1	0	0	1	1	1	1	$a_1 \vee (a_2 \wedge a_3)$
$\overrightarrow{\mu^L}_{a_2 \vee (a_1 \wedge a_3)}$	0	1	0	1	1	1	1	$a_2 \vee (a_1 \wedge a_3)$
$\overrightarrow{\mu^L}_{a_3 \vee (a_1 \wedge a_2)}$	0	0	1	1	1	1	1	$a_3 \vee (a_1 \wedge a_2)$
$\overrightarrow{\mu^L}_{a_1 \vee a_2}$	1	1	0	1	1	1	1	$(a_1 \vee a_2)$
$\overrightarrow{\mu^L}_{a_1 \vee a_3}$	1	0	1	1	1	1	1	$(a_1 \vee a_3)$
$\overrightarrow{\mu^L}_{a_2 \vee a_3}$	0	1	1	1	1	1	1	$(a_2 \vee a_3)$
$\overrightarrow{\mu^L}_{a_1 \vee a_2 \vee a_3}$	1	1	1	1	1	1	1	$(a_1 \vee a_2 \vee a_3)$

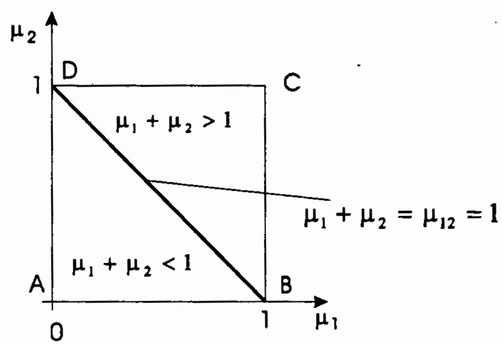

Figure 1: Fuzzy measures for two attributes

3.5 Any fuzzy measure and logical fuzzy measures

Since the Choquet integral for any logical fuzzy measure has a clear logical representation, the relations between any fuzzy measure (or a fuzzy measure vector) and fuzzy logical measures (or logical fuzzy measure vectors) are important.

Proposition 2 *(A) Every linear convex combination of logical fuzzy measure vectors $\overrightarrow{\mu_q^L}$, $q = 1, ..., Q$ is a fuzzy measure vector, and*
(B) Any fuzzy measure vector, $\overrightarrow{\mu}$, can be represented as a linear convex combination of logical fuzzy measure vectors $\overrightarrow{\mu_q^L}$, $q = 1, ..., Q$.

$$\overrightarrow{\mu} = \sum_{q=1}^{Q} \lambda_q \overrightarrow{\mu_q^L}$$

where $\sum_{q=1}^{Q} \lambda_q = 1$ and $\lambda_q \geq 0$, $q = 1, ..., Q$.

Proof. (A) Since from the definition of a logical fuzzy measure:

$$\mu_q^L(\emptyset) = 0, \qquad q = 1, ..., Q$$

$$\mu_q^L(\Omega) = 1, \qquad q = 1, ..., Q$$

the boundary conditions of any linear convex combination of logical fuzzy measures are:

$$\begin{aligned} \mu(\emptyset) &= \sum_{q=1}^{Q} \lambda_q \mu_q^L(\emptyset) = 0 \\ \mu(\Omega) &= \sum_{q=1}^{Q} \lambda_q \mu_q^L(\Omega) = 1. \\ \sum_{q=1}^{Q} \lambda_q &= 1 \quad \lambda_q \geq 0, \quad q = 1, ..., Q. \end{aligned}$$

and by definition of fuzzy logical measures for $A \subset B$

$$\mu_q^L(A) \leq \mu_q^L(B), \qquad q = 1, ..., Q$$

monotonicity condition for any linear convex combination of logical fuzzy measures is satisfied:

$$\begin{aligned} \mu(A) &= \sum_{q=1}^{Q} \lambda_q \mu_q^L(A) \leq \sum_{q=1}^{Q} \lambda_q \mu_q^L(B) = \mu(B) \\ \sum_{q=1}^{Q} \lambda_q &= 1 \quad \lambda_q \geq 0, \quad q = 1, ..., Q. \end{aligned}$$

and it follows that any linear convex combination of logical fuzzy measure is a fuzzy measure.

(B) The components of any fuzzy measure vector

$$\begin{aligned}\overrightarrow{\mu} &= [\mu(\emptyset), ..., \mu(\Omega)]^T \\ &= [\mu_1, ..., \mu_{2^n}]^T\end{aligned}$$

can be permuted as follows:

$$1 = \mu_{(1)} \geq ... \geq \mu_{(2^n)} = 0,$$

and after choosing only mutually different values

$$1 = \mu_{(1)} > ... > \mu_{(Q)} > 0,$$

where $Q \leq 2^n$, then $\overrightarrow{\mu}$ can be written as:

$$\begin{aligned}\overrightarrow{\mu} &= \left(\mu_{(1)} - \mu_{(2)}\right) \overrightarrow{\mu}_1^L + \\ &\quad \left(\mu_{(2)} - \mu_{(3)}\right) \overrightarrow{\mu}_2^L + \\ &\quad \vdots \\ &\quad \left(\mu_{(Q)} - 0\right) \overrightarrow{\mu}_Q^L,\end{aligned}$$

where:

$$\begin{aligned}\mu_1^L(A) &= \begin{cases} 1, & \mu(A) = 1 \\ 0, & \mu(A) < 1 \end{cases}, \quad A \subseteq \Omega, \\ \mu_2^L(A) &= \begin{cases} 1, & \mu(A) \geq \mu_{(2)} \\ 0, & \mu(A) < \mu_{(2)} \end{cases}, \quad A \subseteq \Omega, \\ &\vdots \\ \mu_q^L(A) &= \begin{cases} 1, & \mu(A) \geq \mu_{(q)} \\ 0, & \mu(A) < \mu_{(q)} \end{cases}, \quad A \subseteq \Omega, \\ &\vdots \\ \mu_Q^L(A) &= \begin{cases} 1, & \mu(A) > 0 \\ 0, & \mu(A) = 0 \end{cases}, \quad A \subseteq \Omega;\end{aligned}$$

and $\overrightarrow{\mu}_q^L$ for $q = 1, ...Q$ are logical fuzzy measure vectors, since from the definition it follows that if $\mu_q^L(A) = 1$ then $\forall B \supseteq A$ is $\mu_q^L(B) = 1$ as a consequence of the definition of a fuzzy measure

$$\mu_q^L(B) \geq \mu_q^L(A) \Rightarrow \mu_q^L(B) \geq \mu_{(q)} \Rightarrow \mu_q^L(B) = 1.$$

If we introduce

$$\lambda_q := \mu_{(q)} - \mu_{(q+1)} > 0$$

the above expression for a fuzzy measure vector is:

$$\overrightarrow{\mu} = \sum_{q=1}^{Q} \lambda_q \overrightarrow{\mu}_q^L$$

where:

$$\sum_{i=1}^{Q} \lambda_q = \mu_{(1)} - \mu_{(2)} + \mu_{(2)} - \mu_{(3)} + ... + \mu_{(Q)} - 0$$
$$= \mu_{(1)} = 1.$$

So any fuzzy measure vector can be represented as a linear convex combination of logical fuzzy measure vectors. ■

Example 9 *An arbitrary fuzzy measure vector $\vec{\mu}$ for two attributes can be represented by logical fuzzy measure vectors $\vec{\mu_w^L}$, $w \in W = \{a_1 \wedge a_2, a_1, a_2, a_1 \vee a_2\}$; defined in the example 7 as follows:*

$$\vec{\mu} = \sum_{w \in W} \lambda_w \vec{\mu_w^L},$$

where

$$\sum_{w \in W} \lambda_w = 1, \quad \lambda_w \geq 0,$$

and for the special cases:

	$\mu_1 + \mu_2 \geq 1$	$\mu_2 \geq \mu_1$	$\mu_1 + \mu_2 < 1$	$\mu_2 < \mu_1$
$\lambda_{a_1 \wedge a_2}$	*0*	$1 - \mu_2$	$1 - \mu_1 - \mu_2$	$1 - \mu_1$
λ_{a_1}	$1 - \mu_2$	*0*	μ_1	$\mu_1 - \mu_2$
λ_{a_2}	$1 - \mu_1$	$\mu_2 - \mu_1$	μ_2	*0*
$\lambda_{a_1 \vee a_2}$	$\mu_1 + \mu_2 - 1$	μ_1	*0*	μ_2

It is obvious that the representation is not unique.

3.6 Logical representation of the discrete Choquet integral

From propositions: (a.) about any fuzzy measure and logical fuzzy measures and (b.) the linearity of discrete Choquet integral by measures, follows the following proposition.

Proposition 3 *The fuzzy Choquet integral for any fuzzy measure can be represented as a convex combination of logical Choquet integrals.*

$$C\mu(a_1, ..., a_n) = \sum_{q=1}^{Q} \lambda_q C_{\mu_q^L}(a_1, ..., a_n)$$
$$= \sum_{q=1}^{Q} \lambda_q \bigvee_{A:\mu_q^L(A)=1} \left(\bigwedge_{a_k \in A} a_k \right)$$

So, the application of discrete Choquet integral reduces the characteristic interactions between attributes to expressions which contain continuous logical functions of the elements of the power set of attributes. Continuous logical functions are of type OR (defined as max) and AND (defined as min) and their combinations as a reduction to isolated attributes.

Example 10 *The discrete Choquet integral for two attributes and an arbitrary fuzzy measure* $C\mu(a_1, a_2)$ *has the following equivalent extensive representation:*

$$C\mu(a_1,a_2) = \lambda_{a_1 \wedge a_2}(a_1 \wedge a_2) + \lambda_{a_1} a_1 + \lambda_{a_2} a_2 + \lambda_{a_1 \vee a_2}(a_1 \vee a_2)$$

where $\sum_{w \in W} \lambda_w = 1, \quad \lambda_w \geq 0, \quad W = \{a_1 \wedge a_2, a_1, a_2, a_1 \vee a_2\}$. *For the special cases:*

a) redundancy: $\mu_1 + \mu_2 \geq \mu_{12} = 1$. *Since, from the previous example for case of redundance:*

$$\overrightarrow{\mu} = (1 - \mu_2)\overrightarrow{\mu^L_{a_1}} + (1 - \mu_1)\overrightarrow{\mu^L_{a_2}} + (\mu_1 + \mu_2 - 1)\overrightarrow{\mu^L_{a_1 \vee a_2}}$$

and from the property of linearity:

$$C\mu(a_1,a_2) = (1-\mu_2)\, C_{\mu^L_2}(a_1,a_2) + (1-\mu_1)\, C_{\mu^L_3}(a_1,a_2) + (\mu_1+\mu_2-1)\, C_{\mu^L_4}(a_1,a_2)$$

and since:

	μ_1	μ_2	μ_{12}	$C_{\mu_L}(a_1,a_2)$
$\overrightarrow{\mu^L_{a_1}}$	*1*	*0*	*1*	a_1
$\overrightarrow{\mu^L_{a_2}}$	*0*	*1*	*1*	a_2
$\overrightarrow{\mu^L_{a_1 \vee a_2}}$	*1*	*1*	*1*	$a_1 \vee a_2$

it follows that:

$$C\mu(a_1,a_2) = (1-\mu_2)\, a_1 + (1-\mu_1)\, a_2 + (\mu_1+\mu_2-1)(a_1 \vee a_2).$$

In the same way, for: b) $\mu_2 \geq \mu_1$:

$$C\mu(a_1,a_2) = (1-\mu_2)(a_1 \wedge a_2) + (\mu_2 - \mu_1)\, a_2 + \mu_1 (a_1 \vee a_2)$$

c) positive interaction: $\mu_1 + \mu_2 < \mu_{12} = 1$

$$C\mu(a_1,a_2) = \mu_1 a_1 + \mu_2 a_2 + (1 - \mu_1 - \mu_2)(a_1 \wedge a_2)$$

and d) $\mu_2 < \mu_1$:

$$C\mu(a_1,a_2) = (1-\mu_1)(a_1 \wedge a_2) + (\mu_1 - \mu_2)\, a_1 + \mu_2 (a_1 \vee a_2).$$

It is obvious that the representation is not unique.

4 MADM and the Logical Representation of the Discrete Choquet Integral

The logical representation of the Choquet integral in multi-attribute decision making (MADM) is illustrated in this section. A general MADM problem can be reduced to two steps: (a) preprocessing of the initial values of attributes - normalizations, and (b) aggregation of normalized attributes values. Sometimes, a fuzzy measure and a fuzzy integral can be used for aggregation in the presence of interactions between attributes [6]. In the following simple example are given the application of the Choquet integral and its logical representation to MADM.

Example 11 *Alternatives A, B and C are described by two attributes. The values of attributes are given in the following table:*

	a_1	a_2
A	1.1	0.1
B	0.6	0.6
C	0.1	1.1

Alternatives should be arranged (ordered) based on the following two partial requirements: (1.) attribute a_1 is more important than a_2, and (2.) attributes a_1 and a_2 are important simultaneously.
(a) In the fuzzy measure approach, these two requirements reduce to the following two conditions:

$$\begin{array}{ll} \text{1. requirement} \Rightarrow & \mu_1 > \mu_2 \\ \text{2. requirement} \Rightarrow & \mu_1 + \mu_2 < \mu_{12} = 1 \end{array}$$

For example, the following fuzzy measure: $\mu_1 = 1/3$; $\mu_2 = 1/6$; *and* $\mu_{12} = 1$; *satisfies the requirements, and after application of the Choquet integral, we obtain the following ranking of alternatives:*

rang	alternative	$C\mu\,(a_1, a_2)$
1.	B	0.600
2.	A	0.433
3.	C	0.266

(b) In the approach based on the logical representation of the Choquet integral, we have the following aggregation function:

$$\begin{aligned} C\mu\,(a_1, a_2) &= \lambda_1 a_1 + \lambda_2 a_2 + \lambda_3\,(a_1 \wedge a_2) \\ \lambda_i &\geq 0, i = 1, 2, 3; \\ \lambda_1 &> \lambda_2;\ \lambda_1 + \lambda_2 + \lambda_3 = 1, \end{aligned}$$

so the problem is reduced to a classical problem of determination of weighted coefficients but the list of attributes is extended by a new partial requirement - ("generalized attribute") - the logical expression $a_1 \wedge a_2$. *If we choose* $\lambda_1 = 1/3, \lambda_2 = 1/6$, *and* $\lambda_3 = 1/2$ *for weighted coefficients, then we have the same result.*

	a_1	a_2	$a_1 \wedge a_2$	$C\mu\,(a_1, a_2)$
A	1.1	0.1	0.1	0.433
B	0.6	0.6	0.6	0.6
C	0.1	1.1	0.1	0.266
	$\lambda_1 = 1/3$	$\lambda_2 = 1/6$	$\lambda_3 = 1/2$	

Definition 8 *Partial requirements ("generalized attributes") are all or some attributes (chosen attributes)* $(a_1, a_2, ...)$ *and/or logical expressions on attributes (corresponding logical expressions on attributes)* $(a_1 \wedge a_1, ...)$.

As a consequence, to every partial requirement ("generalized attribute") corresponds a logical fuzzy measure.

Definition 9 *A logical fuzzy measure requirement vector is a logical fuzzy measure vector which corresponds to the analyzed partial requirement ("generalized attribute").*

Example 12 *From the previous example: partial requirements ("generalized attributes") and corresponding logical fuzzy measure vector requirements are given in the following table:*

$C\mu\,(a_1, a_2)$		μ_1^L	μ_2^L	μ_{12}^L
a_1	$\overrightarrow{\mu}_{a_1}$	*1*	*0*	*1*
a_2	$\overrightarrow{\mu}_{a_2}$	*0*	*1*	*1*
$a_1 \wedge a_2$	$\overrightarrow{\mu}_{a_1 \wedge a_2}$	*0*	*0*	*1*

The fuzzy measure vector of the resulting requirements, $\overrightarrow{\mu}$*, is:*

$$\begin{aligned} \overrightarrow{\mu} &= \lambda_1 \overrightarrow{\mu}_{a_1} + \lambda_2 \overrightarrow{\mu}_{a_2} + \lambda_3 \overrightarrow{\mu}_{a_1 \wedge a_2} \\ \sum_{i=1}^{3} \lambda_i &= 1; \quad \lambda_j \geq 0, \quad j = 1,2,3; \quad \lambda_1 > \lambda_2, \end{aligned}$$

or

$$\begin{bmatrix} \mu_1 \\ \mu_2 \\ \mu_{12} \end{bmatrix} = \lambda_1 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + \lambda_2 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + \lambda_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

and for weighted coefficients $\lambda_1 = 1/3$*;* $\lambda_2 = 1/6$*; and* $\lambda_3 = 1/2$*, the fuzzy measure of resulting requirements is* $\mu_1 = 1/3$*;* $\mu_2 = 1/6$*; and* $\mu_{12} = 1$.

Aggregation in MADM by the logical representation of the Choquet integral is reduced to a linear convex combination of partial requirements ("generalized attributes") and/or determination of weight coefficients of partial requirements.

A fuzzy measure (vector) of resulting requirements is a linear convex combination of logical fuzzy measure requirement vectors.

Definition 10 *The space of consistent fuzzy measure requirement vectors is a fuzzy measure subspace, generated by linear convex combinations of logical fuzzy measure requirement vectors (vectors of generalized attributes).*

One of the main advantages of the logical representation of the discrete Choquet integral in MADM is the possibility of consistent explanation of a DM's preference structure. This will be illustrated by the example of the evaluation of students according to marks in mathematics, physics and literature [7].

Example 13 *[7] The director of a high school has to evaluate his students according to their level in mathematics (M), physics (P) and literature (L), and he considers the following three students (marks are given on a scale from 0 to 20).*

Student	M	P	L
	a_M	a_P	a_L
A	18	16	10
B	10	12	18
C	14	15	15

The director thinks that: (1.) Scientific subjects (M, P) are more important. (2.) M and P are more or less similar, and students good at M are in general also good at P, so that students good at both must not be favored. (3.) Students good at M (or P) and literature are rather uncommon and must be favored. A consistent resulting fuzzy measure vector $\overrightarrow{\mu}$ must be in the fuzzy subspace - requirement space, defined as a linear convex combination of fuzzy vectors which correspond to partial requirements. The first requirement is actually the subspace defined by the following

$$\lambda'_M \overrightarrow{\mu}^L_M + \lambda'_P \overrightarrow{\mu}^L_P + \lambda'_L \overrightarrow{\mu}^L_L, \quad \lambda'_M = \lambda'_P > \lambda'_L, \quad \lambda'_M + \lambda'_P + \lambda'_L = 1,$$
$$\lambda'_M > 0, \lambda'_P > 0, \lambda'_L > 0.$$

where:

	μ_M	μ_P	μ_L	μ_{MP}	μ_{ML}	μ_{PL}	μ_{MPL}	$C_{\mu_L}(a_M, a_P, a_L)$
$\overrightarrow{\mu}^L_M$	1	0	0	1	1	0	1	a_M
$\overrightarrow{\mu}^L_P$	0	1	0	1	0	1	1	a_p
$\overrightarrow{\mu}^L_L$	0	0	1	0	1	1	1	a_L

The second and third requirement are defined by the following fuzzy logical vectors:

	μ_M	μ_P	μ_L	μ_{MP}	μ_{ML}	μ_{PL}	μ_{MPL}	$C_{\mu_L}(a_M, a_P, a_L)$
$\overrightarrow{\mu}^L_{M\vee P}$	1	1	0	1	1	1	1	$a_M \vee a_P$
$\overrightarrow{\mu}^L_{(M\vee P)\wedge L}$	0	0	0	0	1	1	1	$(a_M \vee a_P) \wedge a_L$

So, the consistent requirement fuzzy subspace is defined by the following linear convex combination:

$$\begin{aligned} \overrightarrow{\mu} &= \lambda_M \overrightarrow{\mu}^L_M + \lambda_P \overrightarrow{\mu}^L_P + \lambda_L \overrightarrow{\mu}^L_L + \lambda_{M\vee P} \overrightarrow{\mu}^L_{M\vee P} + \lambda_{(M\vee P)\wedge L} \overrightarrow{\mu}^L_{(M\vee P)\wedge L} \\ \sum_{w\in W} \lambda_w &= 1, \ \lambda_w \geq 0, \qquad W = \{M, P, L, M\vee P, (M\vee P)\wedge L\}. \end{aligned}$$

It follows that the fuzzy measure requirement can be expressed as a function of weighted coefficients of partial requirements:

$$\begin{bmatrix} \mu_M \\ \mu_P \\ \mu_L \\ \mu_{MP} \\ \mu_{ML} \\ \mu_{PL} \\ \mu_{MPL} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix} * \begin{bmatrix} \lambda_M \\ \lambda_P \\ \lambda_L \\ \lambda_{M\vee P} \\ \lambda_{(M\vee P)\wedge L} \end{bmatrix},$$

where:

$$\sum_{w\in W}\lambda_w = 1; \qquad \lambda_w \geq 0, \qquad w \in W = \{M, P, L, M \vee P, (M \vee P) \wedge L\}.$$

These requirements are by [16] directly expressed as the following fuzzy measure: $\mu_M = \mu_P = 0.45$, $\mu_L = 0.30$, $\mu_{MP} = 0.5, \mu_{ML} = 0.9, \mu_{PL} = 0.9, \mu_{MPL} = 1$. *But, this fuzzy measure is outside of the fuzzy measure subspace of the consistent requirements!*

The logical representation of the discrete Choquet integral for the consistent fuzzy measure requirement is

$$\begin{aligned} C\mu\,(a_M, a_P, a_L) &= \lambda_M a_M + \lambda_P a_P + \lambda_L a_L + \lambda_{M\vee P}(a_M \vee a_P) \\ &\quad + \lambda_{(M\vee P)\wedge L}((a_M \vee a_P) \wedge a_L) \\ \textstyle\sum_{w\in W}\lambda_w = 1,\ \lambda_w \geq 0, &\qquad W = \{M, P, L, M \vee P, (M \vee P) \wedge L\}. \end{aligned}$$

Objects of further decisions could be only the values of weights for subjects M, P and L λ_M, λ_P, *and* λ_L, *redundancy of simultaneous success in M - P* $\lambda_{M\vee P}$, *interaction of simultaneous success in M or P and L* $\lambda_{(M\vee P)\wedge L}$. *It is obvious that it is not possible to find such values for weights that give as a result the fuzzy measure proposed in [7]. If the proposed relation 3:3:2 is kept for M, P and L and if these interactions are of the same importance, and all interactions and all subjects are also of the same importance, then the following values could be accepted for weights:* $\lambda_M = \lambda_P = 3/16$, $\lambda_L = 1/8$, $\lambda_{(M\vee P)} = \lambda_{(M\vee P)\wedge L} = 1/4$. *The corresponding fuzzy measure is:* $\mu_M = 7/16$; $\mu_P = 7/16$; $\mu_L = 2/16$; $\mu_{MP} = 17/16$; $\mu_{ML} = 13/16$; $\mu_{PL} = 13/16$; $\mu_{MPL} = 1$, *implying the following results:*

Student	*M*	*P*	*L*	*MorP*	*(MorP)andL*	
a_i	a_M	a_P	a_L	$a_M \vee a_P$	$(a_M \vee a_P) \wedge a_M$	$\sum_i w_i a_i$
A	*18*	*16*	*10*	*18*	*10*	*13.5000*
B	*10*	*12*	*18*	*12*	*12*	*12.1875*
C	*14*	*15*	*15*	*15*	*15*	*14.8125*
w_i	*3/16*	*3/16*	*2/16*	*4/16*	*4/16*	

5 Logical Fuzzy Measures and Two Other Measure Representations

There are two other representations of a measure: (a) the Möbius and (b) the interaction representation. Here are introduced a logical Möbius transform for a logical fuzzy measure and a logical interaction index.

5.1 The logical Möbius transform

Definition 11 *The Möbius transform of a fuzzy measure* μ, *is a set function on* Ω *defined by*

$$m(A) := \sum_{B\subset A} (-1)^{|A\setminus B|}\,\mu(B), \ \forall A \subset \Omega.$$

The transformation is invertible, and μ can be recovered from m by

$$\mu(A) = \sum_{B \subset A} m(B), \ \forall A \subset \Omega.$$

The Möbius transform of a fuzzy measure on a finite set can be represented as a Möbius vector $\vec{m}$, with 2^n components, where $n = |\Omega|$, (or $2^n - 1$ since component $m(\emptyset) = 0$ is trivial).

Definition 12 *The components of the Möbius vector, of a finite set of attributes, are the Möbius coefficients of the elements of the power set $\mathcal{P}(\Omega)$.*

Definition 13 *A space defined by all possible Möbius vectors is a Möbius vector (measure) space.*

Definition 14 *Logical Möbius coefficients $m^L(A)$, $\forall A \subset \Omega$, are obtained from the Möbius transform of logical fuzzy measures μ^L.*

Example 14 *The values of logical Möbius coefficients - components of Möbius vectors, for logical fuzzy measures, in the case of two attributes, are given in the following table:*

	m_1	m_2	m_{12}	$C\mu(a_1, a_2)$
$\vec{m^L}_{a_1 \wedge a_2}$	0	0	1	$a_1 \wedge a_2$
$\vec{m^L}_{a_1}$	1	0	0	a_1
$\vec{m^L}_{a_2}$	0	1	0	a_2
$\vec{m^L}_{a_1 \vee a_2}$	1	1	-1	$a_1 \vee a_2$

or, in the Möbius vector space (Fig.2).

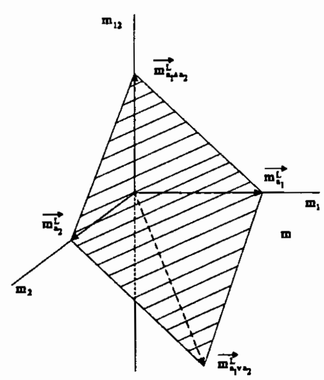

Figure 2: Möbius vector space

Example 15 *Möbius coefficients for logical fuzzy measures, in the case of three attributes, are given in the following table:*

	m_1	m_2	m_3	m_{12}	m_{13}	m_{23}	m_{123}	$C\mu(a_1,a_2,a_3)$
$\overrightarrow{m^L}_{a_1\wedge a_2\wedge a_3}$	0	0	0	0	0	0	1	$(a_1\wedge a_2\wedge a_3)$
$\overrightarrow{m^L}_{a_1\wedge a_2}$	0	0	0	1	0	0	0	$(a_1\wedge a_2)$
$\overrightarrow{m^L}_{a_1\wedge a_3}$	0	0	0	0	1	0	0	$(a_1\wedge a_3)$
$\overrightarrow{m^L}_{a_2\wedge a_3}$	0	0	0	0	0	1	0	$(a_2\wedge a_3)$
$\overrightarrow{m^L}_{a_1\wedge(a_2\vee a_3)}$	0	0	0	1	1	0	-1	$a_1\wedge(a_2\vee a_3)$
$\overrightarrow{m^L}_{a_2\wedge(a_1\vee a_3)}$	0	0	0	1	0	1	-1	$a_2\wedge(a_1\vee a_3)$
$\overrightarrow{m^L}_{a_3\wedge(a_1\vee a_2)}$	0	0	0	0	1	1	-1	$a_3\wedge(a_1\vee a_2)$
$\overrightarrow{m^L}_{\vee(a_i\wedge a_j)}$	0	0	0	1	1	1	-2	$\bigvee_{i,j=1,2,3}(a_i\wedge a_j)$
$\overrightarrow{m^L}_{a_1}$	1	0	0	0	0	0	0	a_1
$\overrightarrow{m^L}_{a_2}$	0	1	0	0	0	0	0	a_2
$\overrightarrow{m^L}_{a_3}$	0	0	1	0	0	0	0	a_3
$\overrightarrow{m^L}_{a_1\vee(a_2\wedge a_3)}$	1	0	0	0	0	1	-1	$a_1\vee(a_2\wedge a_3)$
$\overrightarrow{m^L}_{a_2\vee(a_1\wedge a_3)}$	0	1	0	0	1	0	-1	$a_2\vee(a_1\wedge a_3)$
$\overrightarrow{m^L}_{a_3\vee(a_1\wedge a_2)}$	0	0	1	1	0	0	-1	$a_3\vee(a_1\wedge a_2)$
$\overrightarrow{m^L}_{a_1\vee a_2}$	1	1	0	-1	0	0	0	$(a_1\vee a_2)$
$\overrightarrow{m^L}_{a_1\vee a_3}$	1	0	1	0	-1	0	0	$(a_1\vee a_3)$
$\overrightarrow{m^L}_{a_2\vee a_3}$	0	1	1	0	0	-1	0	$(a_2\vee a_3)$
$\overrightarrow{m^L}_{a_1\vee a_2\vee a_3}$	1	1	1	-1	-1	-1	1	$(a_1\vee a_2\vee a_3)$

Proposition 4 *Möbius coefficients for an arbitrary measure μ can be expressed as a convex combination of corresponding logical Möbius coefficients $m_q^L(A)$, $\forall A\subset\Omega$, $q=1,...,Q$.*

$$m(A)=\sum_{q=1}^{Q}\lambda_q m_q^L(A).$$

or, in the vector form

$$\vec{m}=\sum_{q=1}^{Q}\lambda_q\overrightarrow{m^L}_q.$$

Proof. Follows from the linearity of m.

Example 16 *An arbitrary Möbius vector $\vec{m}$ for two attributes can be represented by logical Möbius vectors $\overrightarrow{m_w^L}$, $w\in\{a_1\wedge a_2,a_1,a_2,a_1\vee a_2\}$; defined in Example 13 as follows:*

$$\vec{m}=\sum_{w\in W}\lambda_w\vec{m}_w^L,\quad W=\{a_1\wedge a_2,a_1,a_2,a_1\vee a_2\}$$

where

$$\sum_{w \in W} \lambda_w = 1, \quad \lambda_w \geq 0, \quad W = \{a_1 \wedge a_2, a_1, a_2, a_1 \vee a_2\}$$

and for the special cases:

	$m_1 + m_2 \geq 1$	$m_2 \geq m_1$	$m_1 + m_2 < 1$	$m_2 < m_1$
$\lambda_{a_1 \wedge a_2}$	0	$1 - m_2$	$1 - m_1 - m_2$	$1 - m_1$
λ_{a_1}	$1 - m_2$	0	m_1	$m_1 - m_2$
λ_{a_2}	$1 - m_1$	$m_2 - m_1$	m_2	0
$\lambda_{a_1 \vee a_2}$	$m_1 + m_2 - 1$	m_1	0	m_2

5.2 Logical index of importance and interaction index

The interaction index defines the interactions between subsets of attributes.

Definition 15 *[7] Let μ be a fuzzy measure on Ω. The interaction index for subset $A \subset \Omega$ is defined by*

$$I(A) := \sum_{k=0}^{n-|A|} \xi_k^{|A|} \sum_{K \subset \Omega \backslash A, |K|=k} \sum_{P \subset A} (-1)^{|A|-|P|} \mu_{PK}$$

with $\xi_K^p = (n-k-p)!k!/(n-p+1)!$.

In [7] it is shown that special cases of this interaction index are:
(a) For $A = \{i\}$, the Shapley index v_i

$$v_i := \sum_{k=0}^{n-1} \gamma_k \sum_{K \subset \Omega \backslash i, |K|=k} (\mu_{iK} - \mu_K)$$

with $\gamma_k := (n-k-1)!k!/n!$, and
(b) For $A = \{i, j\}$, the interaction index of two attributes I_{ij}

$$I_{ij} := \sum_{k=0}^{n-2} \zeta_k \sum_{\substack{K \subset \Omega \backslash \{i,j\} \\ |K|=k}} (\mu_{ijK} - \mu_{iK} - \mu_{jK} + \mu_K)$$

with $\zeta_k :=.(n-k-2)!k!/(n-1)!$

Any interaction index of a finite set of attributes (players), can be represented as an interaction vector $\overrightarrow{I}$, with 2^n components, where $n = |\Omega|$.

Definition 16 *Interaction vector components of a finite set of attributes are interaction indices of the elements of the power set $\mathcal{P}(\Omega)$.*

Definition 17 *A space defined by all possible interaction vectors is an interaction vector (measure) space.*

Definition 18 *The logical interaction index $I^L(A)$ for subset $A \subset \Omega$ is the interaction index defined by logical fuzzy measure μ_L.*

Example 17 *Values of the logical interaction indexes - components of logical interaction vectors, and corresponding logical interpretation of Choquet integral, in case of two attributes, are given in the following table:*

	I_1^L	I_2^L	I_{12}^L	$C\mu(a_1, a_2)$
$\overrightarrow{I^L}_{a_1 \wedge a_2}$	$\frac{1}{2}$	$\frac{1}{2}$	*1*	$a_1 \wedge a_2$
$\overrightarrow{I^L}_{a_1}$	*1*	*0*	*0*	a_1
$\overrightarrow{I^L}_{a_2}$	*0*	*1*	*0*	a_2
$\overrightarrow{I^L}_{a_1 \vee a_2}$	$\frac{1}{2}$	$\frac{1}{2}$	*-1*	$a_1 \vee a_2$

or, in the interaction vector space (Figure 3.)

Example 18 *Interaction logical vectors - interaction indices for logical fuzzy measures, in the case of three attributes, are given in the following table:*

	$I_{\{1\}}^L$	$I_{\{2\}}^L$	$I_{\{3\}}^L$	$I_{\{1,2\}}^L$	$I_{\{1,3\}}^L$	$I_{\{2,3\}}^L$	$I_{\{1,2,3\}}^L$	$C\mu(a_1, a_2, a_3)$
$\overrightarrow{I}_{a_1 \wedge a_2 \wedge a_3}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	1	$(a_1 \wedge a_2 \wedge a_3)$
$\overrightarrow{I}_{a_1 \wedge a_2}$	$\frac{1}{2}$	$\frac{1}{2}$	0	1	0	0	0	$(a_1 \wedge a_2)$
$\overrightarrow{I}_{a_1 \wedge a_3}$	$\frac{1}{2}$	0	$\frac{1}{2}$	0	1	0	0	$(a_1 \wedge a_3)$
$\overrightarrow{I}_{a_2 \wedge a_3}$	0	$\frac{1}{2}$	$\frac{1}{2}$	0	0	1	0	$(a_2 \wedge a_3)$
$\overrightarrow{I}_{a_1 \wedge (a_2 \vee a_3)}$	$\frac{2}{3}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	-1	$a_1 \wedge (a_2 \vee a_3)$
$\overrightarrow{I}_{a_2 \wedge (a_1 \vee a_3)}$	$\frac{1}{6}$	$\frac{2}{3}$	$\frac{1}{6}$	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	-1	$a_2 \wedge (a_1 \vee a_3)$
$\overrightarrow{I}_{a_3 \wedge (a_1 \vee a_2)}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{2}{3}$	$-\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	-1	$a_3 \wedge (a_1 \vee a_2)$
$\overrightarrow{I}_{\vee(a_i \wedge a_j)}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	0	0	0	-2	$\bigvee_{i,j=1,2,3}(a_i \wedge a_j)$
$\overrightarrow{I}_{a_1}$	1	0	0	0	0	0	0	a_1
$\overrightarrow{I}_{a_2}$	0	1	0	0	0	0	0	a_2
$\overrightarrow{I}_{a_3}$	0	0	1	0	0	0	0	a_3
$\overrightarrow{I}_{a_1 \vee (a_2 \wedge a_3)}$	$\frac{2}{3}$	$\frac{1}{6}$	$\frac{1}{6}$	$-\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	-1	$a_1 \vee (a_2 \wedge a_3)$
$\overrightarrow{I}_{a_2 \vee (a_1 \wedge a_3)}$	$\frac{1}{6}$	$\frac{2}{3}$	$\frac{1}{6}$	$-\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	-1	$a_2 \vee (a_1 \wedge a_3)$
$\overrightarrow{I}_{a_3 \vee (a_1 \wedge a_2)}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{2}{3}$	$\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	-1	$a_3 \vee (a_1 \wedge a_2)$
$\overrightarrow{I}_{a_1 \vee a_2}$	$\frac{1}{2}$	$\frac{1}{2}$	0	-1	0	0	0	$(a_1 \vee a_2)$
$\overrightarrow{I}_{a_1 \vee a_3}$	$\frac{1}{2}$	0	$\frac{1}{2}$	0	-1	0	0	$(a_1 \vee a_3)$
$\overrightarrow{I}_{a_2 \vee a_3}$	0	$\frac{1}{2}$	$\frac{1}{2}$	0	0	-1	0	$(a_2 \vee a_3)$
$\overrightarrow{I}_{a_1 \vee a_2 \vee a_3}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$-\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	1	$(a_1 \vee a_2 \vee a_3)$

Proposition 5 *The interaction index $I(A)$ $\forall A \subset \Omega$, for an arbitrary measure μ can be represented as a convex combination of corresponding logical interaction indices $I_q^L(A)$, $q = 1, ..., Q$.*

$$I(A) = \sum_{q=1}^{Q} \lambda_q I_q^L(A).$$

Proof. *Follows from the linearity of* μ.

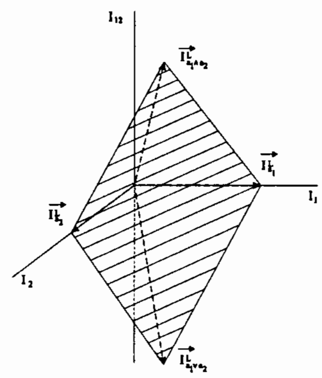

Figure 3: Interaction vector space

Example 19 *An arbitrary interaction vector* $\overrightarrow{I}$ *for two attributes (players) can be represented by logical interaction where interaction vectors,* $\overrightarrow{I^L_w}$,
$w \in \{a_1 \wedge a_2, a_1, a_2, a_1 \vee a_2\}$; *defined in the example 13 as follows:*

$$\overrightarrow{I} = \sum_{w \in W} \lambda_w \overrightarrow{I^L_w}, \quad W = \{a_1 \wedge a_2, a_1, a_2, a_1 \vee a_2\}$$

where

$$\sum_{w \in W} \lambda_w = 1, \quad \lambda_w \geq 0, \quad W = \{a_1 \wedge a_2, a_1, a_2, a_1 \vee a_2\}$$

and for the special cases:

	redundance: $I_{12} \leq 0$	$I_2 \geq I_1$	*positive interaction:* $I_{12} > 0$	$I_2 < I_1$
λ_1	*0*	$I_1 + \frac{1}{2}I_{12}$	I_{12}	$I_2 + \frac{1}{2}I_{12}$
λ_2	$I_1 + \frac{1}{2}I_{12}$	*0*	$I_1 - \frac{1}{2}I_{12}$	$I_1 - I_2$
λ_3	$I_2 + \frac{1}{2}I_{12}$	$I_2 - I_1$	$I_2 - \frac{1}{2}I_{12}$	*0*
λ_4	$-I_{12}$	$I_1 - \frac{1}{2}I_{12}$	*0*	$I_2 - \frac{1}{2}I_{12}$

6 Conclusion

In this paper an equivalent representation, the logical representation, of the discrete Choquet integral is given. The logical representation of the discrete Choquet integral is based on: (a) linearity of the discrete Choquet integral by measures; (b) the logical fuzzy measures with clear interpretation by Choquet integral, and (c) the property that any fuzzy measure can be represented as a convex combination of logical fuzzy measures.

The logical representation consists of a convex combination of logical expressions over the relevant elements of the power set of partial similarities. The logical expressions contain the AND and OR logical operators (defined as *min* and *max*, respectively), and their combinations.

The logical representation of the discrete Choquet integral offers an easy formalization of a desired preference structure in case of MADM. Till now, in practice, it has been very hard to achieve desired characteristics directly by a fuzzy measure [7].

Actually, desired interactions among attributes are defined as the logical expressions and the relative importance of interactions by weighting factors of convex combination. AND logical operator is used for modeling the positive interaction among attributes, and OR for redundance. It is shown that this result is also applicable to the two remaining ways of representing discrete measures: (a) Möbius transformation, and (b) interaction among the subsets of attributes.

Application of the logical representation of discrete Choquet integral to a classification problem in the presence of interaction between attributes will be the subject of a forthcoming paper.

In future work these results will be extended to cases with negative interactions among attributes, based on [15].

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