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Diameter and Adjacency Matrix

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Abstract

New bounds for the diameter of a graph are introduced.
For undirected graphs, two improvements of an old bound are presented. The first one is formulated in terms of the rank of the adjacency matrix, the second one in terms of its eigenvalues.
Further, two known bounds for undirected graphs are extended to directed graphs, by using the minimum polynomial, respectively the rank of the adjacency matrix.

Key words. diameter, adjacency matrix, rank, eigenvalues, minimum polynomial.
AMS subject classification. 05C.

1 Introduction

Any undefined term or notation in this paper may be found in C. Berge [2]. All *graphs* $G = (V, E)$ we consider are finite, without loops or multiple edges, with vertex set V and edge set E. In the first two sections, all graphs considered are undirected. In section 3, we generalize previous notions to directed graphs.

The *adjacency matrix* or vertex-vertex matrix $A(G) = [a_{ij}]$ of a graph G with n vertices is the $n \times n$-matrix which has $a_{ij} = 1$ if the edge (v_i, v_j) exists, and $a_{ij} = 0$ otherwise. The rank $r(G)$ of the adjacency matrix is called the *rank of* G. In the sequel, when this rank is calculated over different fields F, we also denote this rank bij $r_F(G)$.

The *distance* $d(v_i, v_j)$ between two vertices v_i and v_j is the length of a shortest path joining them; if no joining path exists, then $d(v_i, v_j) = \infty$. The *diameter* $\delta(G)$ of a connected graph G is the maximum distance in G.

A path with n vertices is denoted by P_n. Numbering the n vertices consecutively, the adjacency matrix of P_n is of the form

$$A(P_n) = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 & 0 & 0 \\ 1 & 0 & 1 & \dots & 0 & 0 & 0 \\ 0 & 1 & 0 & \dots & 0 & 0 & 0 \\ & \vdots & & \ddots & & \vdots & \\ 0 & 0 & 0 & \dots & 0 & 1 & 0 \\ 0 & 0 & 0 & \dots & 1 & 0 & 1 \\ 0 & 0 & 0 & \dots & 0 & 1 & 0 \end{pmatrix},$$

with eigenvalues $2\cos\frac{1\pi}{n+1}, 2\cos\frac{2\pi}{n+1}, \dots, 2\cos\frac{n\pi}{n+1}$, cf. L. Lovasz, [11, p. 180].

The determination of the diameter of a graph plays a significant role in many problems of operations research, see e.g. D. Bratton [4], A. Ghouila-Houri [9] and M.K. Goldberg [10]. Several bounds on the diameter exist, but almost all of them deal with special classes of graphs.

The following upper bounds for general undirected graphs are known.

The *neighborhood* $\Gamma(x)$ of the vertex $x \in V$ is the set of vertices adjacent to it. A *chain* is a subset $C \subset V$ such that for any two vertices x and y of C, $\Gamma(x) \subset \Gamma(y) \cup \{y\}$ or $\Gamma(y) \subset \Gamma(x) \cup \{x\}$ must hold. The *Dilworth number* $\nabla(G)$ of a graph G, see e.g. R.P.Dilworth [7], is the minimum number of chains covering the vertex set of the graph.

Bound 1 *(S. Foldes and P. Hammer, [8])*
If G is a connected graph with diameter $\delta(G)$ and Dilworth number $\nabla(G)$, then

$$\delta(G) \leq \nabla(G) + 1.$$

Let d_i denote the degree of the vertex v_i. The *Laplacian* of the graph G is defined as the matrix $Q(G) = [q_{ij}]$ where $q_{ii} = d_i$, $q_{ij} = -a_{ij}$ for $i \neq j$ and $[a_{ij}] = A(G)$. If G is a connected graph with maximum degree k, and if λ is the smallest nonzero eigenvalue of the Laplacian $Q(G)$, then Alon and Milman [1] deduced

$$\delta(G) \leq 2\sqrt{2k/\lambda}\,\log_2 n.$$

This bound was improved by F.R.K. Chung [5] and a generalization is given by

Bound 2 *(F. Chung, V. Faber and T. Manteuffel, [6])*
If G is a connected graph with diameter $\delta(G)$, then

$$\delta(G) \leq \lfloor \frac{\cosh^{-1}(n-1)}{\cosh^{-1}((\lambda_n + \lambda_2)/(\lambda_n - \lambda_2))} \rfloor + 1$$

where $0 \leq \lambda_2 \leq \ldots \leq \lambda_n$ are the eigenvalues of the Laplacian of the graph G and $\lfloor . \rfloor$ is the floor function.

N. Biggs [3, p. 13] and L. Lovász [11, p. 73] noted a bound in terms of eigenvalues.

Bound 3 *The number k of distinct eigenvalues of the adjacency matrix of a connected graph G is greater than its diameter $\delta(G)$, or*

$$\delta(G) \leq k - 1.$$

In 1976 we stated in [12] the following upper bound in terms of the rank of the adjacency matrix.

Bound 4 *Let G be a connected graph with diameter $\delta(G)$. If the rank of G is calculated over $\mathbb{R}$, then*

$$\begin{cases} \delta(G) \leq r(G), & \text{if } \delta(G) \in 2\mathbb{N}, \\ \delta(G) \leq r(G) - 1, & \text{if } \delta(G) \notin 2\mathbb{N}. \end{cases}$$

Moreover $\delta(G) = r(G)$ for $\delta(G)$ is even and $\delta(G) = r(G) - 1$ for $\delta(G)$ odd if and only if for all vertices v_i, v_j with the same distance to one of the initial vertices of a path of length $\delta(G)$, we have $\Gamma(v_i) = \Gamma(v_j)$.

By taking examples, one can see that each of the foregoing bounds is attained for some special graph, thus they are all of the best possible type. However, none of them is dominating an other, i.e. none of the bounds is systematically better than the others.

2 Bounds for the diameter of an undirected graph

In the sequel, we give two improvements of Bound 4. The first one is formulated in terms of the rank of the adjacency matrix, the second one in terms of its eigenvalues.

Theorem 2.1 *Let G be a connected graph of diameter $\delta(G)$. If the rank of G is calculated over $\mathbb{Z}_p$, (the ring of integers modulo p), with p prime, then*

$$\begin{cases} \delta(G) \leq r(G), & \text{if } \delta(G) \in 2\mathbb{N}, \\ \delta(G) \leq r(G) - 1, & \text{if } \delta(G) \notin 2\mathbb{N}. \end{cases}$$

Proof : Let P_l be a path in G of length $l - 1 = \delta(G)$. To calculate the rank of the adjacency matrix $A(P_l)$, we perform some elementary row operations on this adjacency matrix. Let $E_{i,j;\lambda}$ denote the replacement of row i by 'row i + λ row j'. First move all columns of $A(P_l)$ one position to the left (such that the original first column now becomes the last column). After the elementary row operations $E_{3,1;(p-1)}$, $E_{4,2;(p-1)}$, ..., $E_{l,l-2;(p-1)}$, the upper triangular matrix

$$\begin{pmatrix} 1 & & & & & & & \\ 0 & 1 & & & & & & \\ 0 & 0 & 1 & & & & & \\ 0 & 0 & 0 & 1 & & & & \\ \vdots & \vdots & \vdots & \vdots & \ddots & & & \\ 0 & 0 & 0 & 0 & \dots & 1 & & \\ 0 & 0 & 0 & 0 & \dots & 0 & 1 & \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 & \alpha_1 \end{pmatrix}$$

is obtained.
If $\delta(G) = l - 1 \in 2\mathbb{N}$, then $\alpha_1 = 0$ and the rank of $A(P_l)$ is $l - 1$. Hence $\delta(G) = l - 1 = r(P_l)$.
If $\delta(G) = l - 1 \notin 2\mathbb{N}$, then $\alpha_1 = (p-1)^i$, with $i = l/2 - 1$ and the rank of $A(P_l)$ equals l. In this case, $\delta(G) = l - 1 = r(P_l) - 1$.

As the longest path P_l is an induced subgraph of the graph G, the adjacency matrix $A(P_l)$ is a submatrix of $A(G)$. Hence $r(P_l) \leq r(G)$ which proves the theorem. □

Remark 2.2 Linear dependence over $\mathbb{R}$ implies linear dependence over $\mathbb{Z}_p$, hence $r_{\mathbb{Z}_p}(G) \leq r_{\mathbb{R}}(G)$ holds for all graphs G, and the bound of Theorem 2.1 improves Bound 4.
Theorem 2.1 can in particular be useful if the rank is calculated over $\mathbb{Z}_2$. In this case the difference between $r_{\mathbb{R}}(G)$ and $r_{\mathbb{Z}_2}(G)$ can be arbitrarily large, as the following example shows.
Consider the graph G with n vertices ($n \in 3\mathbb{N}$) and adjacency matrix

$$A(G) = \begin{pmatrix} K & B & B & \dots & B \\ B & K & 0 & \dots & 0 \\ B & 0 & K & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ B & 0 & 0 & & K \end{pmatrix},$$

where K is the adjacency matrix of the complete graph K_3 and the matrix B is defined as

$$B = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix}.$$

A short calculation shows that $\delta(G) = n/3+1$, while $r_{\mathbb{R}}(G) = n$ and $r_{\mathbb{Z}_2}(G) = n - n/3$.

We use the notation $N_G(\lambda > 0)$ for the number of positive eigenvalues of the adjacency matrix $A(G)$, and respectively $N_G(\lambda < 0)$, $N_G(\lambda \geq 0)$ and $N_G(\lambda \leq 0)$ for the number of negative, nonnegative, and finally nonpositive eigenvalues of $A(G)$.

Lemma 2.3 *If H is an induced subgraph of G, then $N_H(\lambda > 0) \leq N_G(\lambda > 0)$,
$N_H(\lambda < 0) \leq N_G(\lambda < 0), N_H(\lambda \geq 0) \leq N_G(\lambda \geq 0)$ and $N_H(\lambda \leq 0) \leq N_G(\lambda \leq 0)$.*

Proof : Assume that H has n vertices. We start with the case that H is obtained from G by deleting the vertex v_1 from G. In this case, $A(H)$ is a principal submatrix of $A(G)$, obtained by deleting the first row and column from $A(G)$. Let $\alpha_1 \geq \alpha_2 \geq \dots \geq \alpha_n$ be the eigenvalues of $A(H)$, and

$\beta_1 \geq \beta_2 \geq \ldots \geq \beta_n \geq \beta_{n+1}$ the eigenvalues of $A(G)$. The separation theorem for real symmetric matrices (see e.g. Wilkinson [13]), implies that

$$\beta_1 \geq \alpha_1 \geq \beta_2 \geq \alpha_2 \geq \ldots \geq \alpha_{n-1} \geq \beta_n \geq \alpha_n \geq \beta_{n+1},$$

which explains why, for example $N_H(\lambda > 0) \leq N_G(\lambda > 0)$ if H is obtained by deleting one vertex from G.
For a general subgraph H of G, the above separation theorem can be repeated on the sequence of subgraphs $H \subset H \cup \{v_1\} \subset H \cup \{v_1, v_2\} \subset \ldots \subset G = H \cup \{v_1, \ldots, v_n\}$. The same arguments show that $N_H(\lambda < 0) \leq N_G(\lambda < 0), N_H(\lambda \geq 0) \leq N_G(\lambda \geq 0)$ and $N_H(\lambda \leq 0) \leq N_G(\lambda \leq 0)$. □

Theorem 2.4 *If $\delta(G)$ is the diameter of a connected graph G, then*

$$\begin{cases} \delta(G) & \leq \; 2\, Min[N_G(\lambda > 0), N_G(\lambda < 0)] & \text{if } \delta(G) \in 2\mathbb{N} \\ \delta(G) & \leq \; 2\, Min[N_G(\lambda > 0), N_G(\lambda < 0)] - 1 & \text{if } \delta(G) \notin 2\mathbb{N}. \end{cases}$$

Proof : Let P_l be a path in G of length $l - 1 = \delta(G)$. A closer look at the eigenvalues of $A(P_l)$, mentioned in the introduction, learns that

$$N_{P_l}(\lambda > 0) = N_{P_l}(\lambda < 0) = \tfrac{l}{2} \qquad \text{if } l \in 2\mathbb{N}$$

$$N_{P_l}(\lambda > 0) = N_{P_l}(\lambda < 0) = \tfrac{l-1}{2} \quad \text{if } l \notin 2\mathbb{N},$$

hence

$$\begin{array}{ll} \delta(G) = l - 1 = 2N_{P_l}(\lambda > 0) - 1 = 2N_{P_l}(\lambda < 0) - 1 & \text{if } \delta(G) \notin 2\mathbb{N} \\ \delta(G) = l - 1 = 2N_{P_l}(\lambda > 0) = 2N_{P_l}(\lambda < 0) & \text{if } \delta(G) \in 2\mathbb{N}. \end{array}$$

Finally, apply Lemma 2.3 to the subgraph P_l of G to obtain the result as stated. □

Remark 2.5 If one counts additionally the eigenvalues $\lambda = 0$ of $A(G)$, a similar argument shows that

$$\begin{array}{lll} \delta(G) & \leq & 2\, \text{Min}[N_G(\lambda \geq 0), N_G(\lambda \leq 0)] - 2 \;\; \text{if } \delta(G) \in 2\mathbb{N} \\ \delta(G) & \leq & 2\, \text{Min}[N_G(\lambda \geq 0), N_G(\lambda \leq 0)] - 1 \;\; \text{if } \delta(G) \notin 2\mathbb{N}. \end{array}$$

This bound only improves the bound of Theorem 2.4 in case $\lambda = 0$ is not an eigenvalue of $A(G)$. However, this improvement is at most 2 if $\delta(G) \in 2\mathbb{N}$.

Remark 2.6 As the adjacency matrix of an undirected graph G is symmetric, its rank, calculated over $\mathbb{R}$, is equal to the number of non zero eigenvalues,

$$r_{\mathbb{R}}(G) = N_G(\lambda > 0) + N_G(\lambda < 0),$$

which implies that

$$2\, \text{Min}[N_G(\lambda > 0), N_G(\lambda < 0)] \leq r_{\mathbb{R}}(G).$$

This shows that Theorem 2.4 improves Bound 4.

Example 2.7 To end, we illustrate that the two new bounds, Theorem 2.1 and Theorem 2.4, are not dominated by each other. We assume that the parity of $\delta(G)$ is not known, and thus consider the weakest bound in each theorem. Take for example the graph G with adjacency matrix:

$$A(G) = \begin{pmatrix} 0 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 & 1 & 0 \end{pmatrix}.$$

By Theorem 2.1 with $p = 2$, one obtains that $\delta(G) \leq r_{\mathbb{Z}_2}(G) = 4$, while by Theorem 2.4, $\delta(G) \leq 2\ \mathrm{Min}[N_G(\lambda > 0), N_G(\lambda < 0)] = 2\ \mathrm{Min}[3,3] = 6$, (whereas $\delta(G) = 2$).
On the other hand, the bound of Theorem 2.4 is in particular strong when the difference between $N_G(\lambda > 0)$ and $N_G(\lambda < 0)$ is large. Consider for example $G = K_n$, the complete graph with n vertices, then $\delta(G) \leq 2\ \mathrm{Min}[N_G(\lambda > 0), N_G(\lambda < 0)] = 2\ \mathrm{Min}[1, n-1] = 2$, while $r_{\mathbb{Z}_p}(G) = n - 1$ if $n - 1 = p$ and $r_{\mathbb{Z}_p}(G) = n$ if $n - 1 \neq p$, (whereas $\delta(G) = 1$).

3 Bounds for the diameter of a directed graph

The purpose of this section is to extend the upper bounds for undirected graphs to directed graphs. Note that the diameter of a graph is defined for a *connected* undirected graph. If we use directed graphs, then the diameter is only defined for a *strongly connected* graph.

First of all, the definition of the Dilworth number $\nabla(G)$ can be extended to directed graphs. Unfortunately, a bound similar to Bound 1 does not hold, since there exist directed graphs G with n vertices for which $\delta(G) = n - 1$ and $\nabla(G) = 1$.

Secondly, in [6, p. 450], Bound 2 is extended to strongly connected directed graphs. However, this is only possible in case the in-degree of each vertex equals the out-degree.

Before we extend Bound 3 to strongly connected directed graphs, we recall the definition of the minimum polynomial of a matrix A.
For an n-square matrix A, the determinant $\det(A - \lambda I)$ is a polynomial $\phi(\lambda)$ of degree n in λ, which is known as the *characteristic polynomial* of the matrix A. By the Cayley-Hamilton theorem, every n-square matrix A satisfies its characteristic equation $\phi(A) = 0$. The (unique) monique polynomial $m(\lambda)$ of

minimum degree $s \leq n$ such that $m(A) = 0$ is called the *minimum polynomial* of A.

Theorem 3.1 *Let G be a strongly connected directed graph with diameter $\delta(G)$. If s is the degree of the minimal polynomial of $A(G)$ over $\mathbb{R}$, then*

$$\delta(G) \leq s - 1.$$

Proof : Let $m(\lambda) = \lambda^s + a_{s-1}\lambda^{s-1} + \ldots + a_0 \in \mathbb{R}[\lambda]$ be the minimum polynomial of $A(G)$, then $A(G)$ satisfies the following matrix identity:

$$[A(G)]^s + a_{s-1}[A(G)]^{s-1} + \ldots + a_0 I = 0. \qquad (*)$$

Observe that the entry in position (k, l) of $A(G)^m$ is the number of (v_k, v_l)-walks of length m between the vertices v_k, v_l of G. Assuming $\delta(G) \geq s$, there exist two vertices v_i, v_j at distance $d(v_i, v_j) = s$. The matrix $A(G)^s$ has therefore a non-zero entry in position (i, j), whereas the corresponding (i, j) entries of $A(G)^{s-1}, A(G)^{s-2}, \ldots, A(G), I$ are all zero. This contradicts the matrix identity (*), hence $\delta(G) < s$. □

The equality in the above bound is reached for each $n \geq 3$; for example if G is the *directed cycle* C_n with vertices $v_1, \ldots, v_n$ to which one additional arc (v_3, v_1) is added, then $\delta(G) = s - 1$.

Remark 3.2 It is known that a square matrix A is *diagonalizable*, if and only if its minimal polynomial is $m(\lambda) = \Pi_{i=1}^k(\lambda - \lambda_i)$, where $\lambda_1, \ldots, \lambda_k$ are the distinct eigenvalues of A. Hence, for a diagonalizable matrix A, (in particular for the symmetric adjacency matrix of any undirected graph), the number of distinct eigenvalues is equal to the degree of its minimum polynomial. Thus, for graphs with diagonalizable adjacency matrix, Bound 3 and Theorem 3.1 give the same result. However, Bound 3 is no longer correct in case G is a directed graph with non diagonalizable adjacency matrix. The following example illustrates this fact. Consider the graph G with adjacency matrix

$$A(G) = \begin{pmatrix} C_4 & B \\ B & C_4 \end{pmatrix}$$

where C_4 is the adjacency matrix of the *directed* cycle C_4 and B has all entries 0 except the $(4, 1)$-th entry. The diameter of G is $\delta(G) = 7$, while $A(G)$ has only $k = 5$ distinct eigenvalues, but the degree of the minimum polynomial of $A(G)$ is $s = 8$. Thus, Theorem 3.1 is an extension of Bound 3.

Finally, we extend the results of Bound 4 and Theorem 2.1 to directed graphs. Note that the bound of Theorem 2.4 cannot be extended to directed graphs, as the eigenvalues of a non symmetric adjacency matrix are not necessarily real numbers.

Theorem 3.3 *Let G be a strongly connected directed graph with diameter $\delta(G)$. If the rank of G is calculated over $\mathbb{R}$ or $\mathbb{Z}_p$ with p prime, then*

$$\delta(G) \leq r(G).$$

Proof : Let P_l be a shortest *directed* path in G with maximum length $l-1=\delta(G)$. The submatrix $A(P_l)$ of $A(G)$ is of the form

$$A(P_l) = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 & 0 & 0 \\ * & 0 & 1 & \dots & 0 & 0 & 0 \\ * & * & 0 & \dots & 0 & 0 & 0 \\ & \vdots & & \ddots & & \vdots & \\ * & * & * & \dots & 0 & 1 & 0 \\ * & * & * & \dots & * & 0 & 1 \\ * & * & * & \dots & * & * & 0 \end{pmatrix},$$

and has rank at least $l-1$ (over $\mathbb{R}$ and $\mathbb{Z}_p$). Hence $\delta(G) = l-1 \leq r(P_l) \leq r(G)$. □

The above upper bound can be attained for all $n \geq 3$. For example, for n odd, take $G = P_n$, the undirected path with n vertices, then $\delta(P_n) = n-1 = r_{\mathbb{R}}(P_n) = r_{\mathbb{Z}_p}(P_n)$. For n even, say $n = 2m$, consider the graph G with adjacency matrix

$$A(G) = \begin{pmatrix} C_m & B \\ B & C_m \end{pmatrix}$$

where C_m is the adjacency matrix of the *directed* cycle C_m and the $m \times m$ submatrix B has all entries 0 except the $(m,1)$-th entry. The graph G is strongly connected and $\delta(G) = 2m-1 = r_{\mathbb{R}}(G) = r_{\mathbb{Z}_p}(G)$.

Remark 3.4 As in the case of undirected graphs (cf. remark 2.2), calculation of the rank $r(G)$ over $\mathbb{Z}_p$ is to be preferred over calculation of $r(G)$ over $\mathbb{R}$. The following example illustrates that the difference between $r_{\mathbb{R}}(G)$ and $r_{\mathbb{Z}_2}(G)$ can be arbitrarily large. Consider the directed graph G with n vertices ($n \in 3\mathbb{N}$) and adjacency matrix

$$A(G) = \begin{pmatrix} K & B & B & \dots & B \\ B & K & 0 & \dots & 0 \\ B & 0 & K & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ B & 0 & 0 & & K \end{pmatrix},$$

where K is the adjacency matrix of the complete graph K_3 and the matrix B is

$$B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

The diameter of G is $\delta(G) = n/3 + 1$, the rank $r_{\mathbb{R}}(G) = n$ but $r_{\mathbb{Z}_2}(G) = n - n/3$.

Example 3.5 Finally, we illustrate that the bound $\delta(G) \leq s - 1$ is not dominated by the bound $\delta(G) \leq r(G)$ and vice versa.
Consider the graph G with adjacency matrix

$$A(G) = \begin{pmatrix} 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 1 & 0 \end{pmatrix},$$

which has minimum polynomial $m(\lambda) = \lambda^5 - 6\lambda^3 - 8\lambda^2 - 2\lambda$.
Theorem 3.1 states that $\delta(G) \leq s - 1 = 5 - 1 = 4$, while Theorem 3.3 improves this result to $\delta(G) \leq r_{\mathbb{Z}_2}(G) = 3$, (whereas $\delta(G) = 2$).
On the other hand, consider G with adjacency matrix

$$A(G) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix},$$

which has minimum polynomial $m(\lambda) = \lambda^4 - \lambda - 1$.
Theorem 3.3 implies $\delta(G) \leq r_{\mathbb{Z}_2}(G) = 4$, and by Theorem 3.1 one obtains $\delta(G) \leq s - 1 = 4 - 1 = 3$, (whereas $\delta(G) = 3$).

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