DESCRIPTION OF MEDIAN GAME THEORY WITH EXAMPLES OF COMPETITIVE AND MEDIAN COMPETITIVE GAMES

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Abstract.

Random selection of strategies greatly extends the opportunity to develop optimum strategies for discrete two-person games. A consequence, however, is that the payoffs received by the players can have probability distributions, which complicates the determination of optimum strategies. This problem can be greatly simplified by only considering some reasonable type of « representative value » for a distribution. The expected-value approach uses the distribution mean. The distribution median is another reasonable possibility. For the common situation where the players behave competitively, a form of game theory is developed by applying the median approach to the payoffs for each player. This form of median game theory has very desirable properties with respect to effort needed for application and, compared to expected-value game theory, strong advantages with respect to generality of application. For example, the payoffs can be of a very general nature. A player has an optimum strategy when the game is one player median competitive (OPMC) for him. A game is median competitive when it is OPMC for both players. Competitive games are an important subclass of median competitive games wherein nondecreasing desirability of the payoffs for one player corresponds to nonincreasing desirability of the payoffs to the other player. This paper contains an introduction to median game theory and examples of competitive, OPMC for one player, and median competitive games.

Introduction and discussion.

Two players, each with choice among a finite number of stategies, is the situation considered. Each player selects one of his strategies, separately and independently of the choice made by the other player. A pair of payoffs, one to each player, is associated with every possible combination of a strategy choice by each player. These pairs of payoffs are the possible outcomes for the game. Statement of the possible payoffs to a player in matrix form is con-

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venient, where the rows represent his strategies and the columns the strategies of the other player. Both of the payoff matrices are known to the two players.

A player is said to use a mixed strategy when he assigns probabilities (sum to uniy) to his possible strategies and randomly selects the strategy to be used according to these probabilities. The payoff to each player has a probability distribution (determined by the probabilities that the players assign) when at least one player selects his strategy randomly. Knowledge of the probability distributions of the payoffs is the maximum information that possibly can be obtained about the payoffs occurring for a game.

Determination of optimum mixed strategies is a basic problem of game theory. That is, the problem is to optimally choose the probabilities for the mixed strategies, where unit probabilities are possible. Unfortunately, many complications cloud this choice when all the properties of distributions receive consideration. The problem is greatly simplified, however, when consideration is limited to some kind of « representative value » for a distribution. The distribution mean (expected payoff to the player) is used as the representative value in the well established expected-value approach. Another reasonable way to represent a distribution of payoffs is by its median, and this is the basis for median game theory.

One form of median game theory is that where the payoff matrices are considered separately. The payoffs are ranked according to increasing desirability within each matrix and the situation is such that the resulting rankings are the same for both players (that is, the players are in agreement on the rankings). The median approach is applied to the payoffs for the players (with respect to the orderings). This form of median game theory receives virtually all the consideration in this paper. Another form, based on rankings of outcomes, is being developed. However, all publications to date are concerned with rankings of payoffs.

A very desirable feature of median game theory is that the payoff « values » can be of an exceedingly general nature. Some or all of the payoffs need not even be numbers (for example, might designate categories). A ranking of payoffs, within a matrix, should virtually always be possible (for example, on a paired comparison basis). However, the players are required to agree on the rankings.

The payoffs are required to be numbers (ordinarily expressed in the same unit) for expected-value game theory. Moreover these numbers are required to satisfy the arithmetical operations. This excludes for example, the important situation where the payoff values in one or both matrices are ranks.

Another very desirable feature of the median game theory considered here concerns the necessity for accurate evaluation of payoffs. Knowledge of the relative ranking within each matrix, combined with accurately detemined « values » for at most two payoffs in each matrix (whose locations are identified by the rankings) is sufficient for application. Ordinarily, all the payoffs need to be accurately evaluated for expected-value game theory. The effort required for evaluating payoffs can be a very important practical consideration (ref. 1). For example, suppose that each player has 400 strategies, which is not unusually large for meaningful practical situations. Then, the number of combinations of strategies is 160.000. Obtaining enough information to rank 160.000 payoffs usually requires a small fraction of the effort needed to accurately evaluate 160.000 payoffs.

An important class of games is that in which the players behave competitively toward each other. Then, the concepts of a player acting protectively, or vindictively, are helpful in determination of optimum strategies (ref. 2). A protective player attempts to maximize the payoff he receives, regardless of the payoff to the other player. A vindictive player tries to minimize the payoff to the other player, without consideration of his own payoff. A (mixed) strategy whereby a player can be simultaneously protective and vindictive is an optimum strategy for him when the behavior is competitive.

The competitive viewpoint is adopted for the median game theory based on rankings of payoffs. An optimum solution occurs for a player if and only if the game is one-player-median-competitive [OPMC]. A game is median competitive if and only if it is OPMC for both players. Identification of OPMC games is considered in ref. 3. Special cases of median competitive games (competitive games, or generated by a competitive game) are identified in ref. 2. A game is competitive when its outcomes can be arranged in sequence so that the payoffs to one player have nondecreasing desirability and also the payoffs to the other player have nonincreasing desirability.

The situation of competitive behavior also is that considered for expected-value game theory (for example, see ref. 4). Optimum solutions, of a minimax nature, occur for games that satisfy a zero-sum condition (sum of payoffs is zero for all strategy combinations) or some mild modifications of this condition. Such games are a special case of competitive games and a very small subclass of the median competitive games.

Thus, this median game theory has strong application advantages over expected-value game theory, with respect to both generality of application and effort required for application.

Some results for median game theory are stated in the next section. This is followed by some examples of games that are competitive, generated by a competitive game, OPMC for one player only, and median competitive but not generated by a competitive game.

Some median results.

For simplicity in stating results, the desirability of a payoff and the « value » of a payoff are considered to be the same. The referenced developments of results are stated in terms of payoff values, with those values being numbers. However, it is easily seen that these results apply to situations where relative desirability can be determined among the payoffs for each player (and also the players agree on the resulting orderings).

The players are called I and II and, for standardization, the payoff to player I is listed first in a game outcome. In all cases : there is a largest value P_I (P_{II}) in the payoff matrix for player I (II) such that, when acting protectively, he can assure at least this payoff with probability at least 1/2. Also, there is a smallest value P'_I (P'_{II}) in the matrix for player I (II) such that vindictive player II (I) can assure, with probability at least 1/2, that player I (II) receives at most this payoff. The relations $P'_I \leqslant P_I$ and $P'_{II} \leqslant P_{II}$ hold, with equality possible. Detailed methods for determining P_I, P_{II}, P'_I, P'_{II}, also protective median optimum strategies and vindictive median optimum strategies, are given in refs. 2 and 3 (with the method of ref. 3 usually being preferable). An outline of the method in ref. 3 is given in the Appendix.

Games occur such that a player can be simultaneously protective and vindictive. This happens if and only if the game is OPMC for this player. More specifically, let set I (I) be those outcomes where the payoff to player I (II) is at least P_I (P_{II}) and also the payoff to player II (I) is at most P'_{II} (P'_I). A game is OPMC for player I (II) if and only if he can assure, with probability at least 1/2, that an outcome of set I (II) occurs. To determine whether a game is OPMC for player I (II), first mark the payoffs in his matrix that belong to the outcomes of set I (II). Then form a new payoff matrix for player I (II) by replacing the marked payoffs of his matrix with unity and the unmarked payoffs with zero. Consider the resulting matrix of ones and zeroes to be the payoff matrix for player I (II) in a zero-sum game with an expected-value basis and solve for the value of the game to player I (II). The situation is OPMC for player I (II) if and only if this game value is at least 1/2. When this is the case an optimum strategy for player I (II) in solution of this zero-sum game is median optimum for him. Some further discussion is given in the Appendix. A game is median competitve if and only if it is OPMC for both players. The OPMC results are given in ref. 3.

For standardization purposes, a game is considered to be competitive if and only if the totality of its outcomes can be arranged in a sequence so that the payoff values for player I are nondecreasing and also the payoff values to player II are nonincreasing. An important special case is that where the payoffs to player I are strictly monotonic increasing and simultaneously the payoffs to player II are strictly monotonic decreasing.

Now, consider some new material on OPMC games that is given in this paper. On OPMC game for player I (II) is generated by a competitive game when there exists a sequence arrangement of the totality of outcomes such that : First, the payoffs of player I (II) in outcomes that, in the sequence, are above (below) any outcome with payoff P_I (P_{II}) have values at least (most) equal to P_I (P_{II}), and the payoffs in outcomes below (above) any outcome with payoff P_I (P_{II}) are at most (least) equal to P_I (P_{II}). Second, also the payoffs of player II (I) in outcomes above (below) any outcome with payoff P'_{II} (P'_I) are at most (least) equal to P'_{II} (P'_I), and the payoffs in outcomes below (above) any outcome with payoff P'_{II} (P'_I) are at least (most) equal to P'_{II} (P'_I). A median competitive game is generated by a competitive game if and only if it is OPMC generated by a competitive game for both players, which is a case considered in ref. 2.

Competitive games have desirable features when the possibility of cooperation between the players is considered, and some of the median competitive games that are generated by competitive games also have these desirable features (ref. 5). In addition, interpretation of the implications of an optimum median solution is greatly simplified when the game is competitive, and somewhat simplified when the median competitive situation was generated by a competitive game. As will be seen from the examples, a game that is OPMC for a player, or both players, is not necessarily generated by a competitive game.

To summarize, for the form of median game theory considered, an optimum solution exists for a player if and only if the game is OPMC for him. A procedure is outlined for determining whether a game is OPMC for a player, and for determining a median optimum strategy when the game is OPMC for him. Then, when player I (II) uses a median optimum strategy, he assures with probability at least 1/2 that simultaneously he receives at least P_I (P_{II}) and that the other player receives at most P'_{II} (P'_I).

Finally, consider a possible extension to another form of median game theory. Here, the outcomes are ranked, separately by each player, and there need not be any agreement in these rankings. The median approach is applied to these rankings of outcomes. An advantage is almost complete generality of application, with solutions for situations where the players do not behave competitively (or only partially competitively). A disadvantage is the substantial increase in the effort needed for application. Often, all of the payoffs would need to be accurately evaluated. A first step in the development of this form of game theory, for competitive behavior, occurs in ref. 6. The procedure used in ref. 6 is to suitably supplement set I (II) with outcomes until the first time player I (II) can assure an outcome of his augmented set with probability at least 1/2.

Examples.

To illustrate some of the aspects of median game theory, six examples of discrete two-person games are considered. Player I has five strategies and player II has four strategies. For both players, the possible payoffs are the numbers 1(1)20, where these could represent ranks for one or both players.

The examples are selected so that in all cases $P_I = 13$ and $P_{II} = 14$. When the game is OPMC for player I, the relation $P'_I = 7$ holds. When the game is OPMC for player II, the relation $P'_{II} = 8$ holds. The Appendix contains some discussion of cases where P_I, P_{II}, P'_I, P'_{II} and median optimum solutions are readily determined. These considerations receive direct use in obtaining the results that are stated in the following material.

An example of a competitive game occurs for the payoff matrices in Table 1. The twenty possible outcomes can be arranged in sequence so that the payoffs to player I are increasing and the payoffs to player II are decreasing. A median optimum mixed strategy for player I is obtained by assigning probability 1/2 to each of his strategies 2 and 3. For player II, a median optimum strategy is obtained by assigning probability 1/2 to each of his strategies 1 and 2.

The game of Table 2 is generated by the game of Table 1. Here, $P'_I = 7$ and $P'_{II} = 8$. The matrices of Table 2 are obtained by exchanging payoffs within the matrices of Table 1 so that the conditions for generation of a median competitive game are satisfied. The median optimum strategies for the game of Table 1 are also optimum for the game of Table 2.

Table 3 contains a game that is OPMC for player I and, for him, is generated from the competitive game of Table 1. That is, all payoffs at least equal to $P_I = 13$ for player I are paired with payoffs at most equal to $P'_{II} = 8$ for player II. The game is not OPMC for player II in any sense. Examination shows that $P_{II} = 14$ and $P'_{II} = 6$. Let the outcomes where the payoff is at least 14 to player II and also the payoff to player I is at most 6 be marked in the matrix for player II. An outcome of this marked set cannot be assured with probability at least 1/2 by player II. As before, a median optimum strategy for player I consists in randomly selecting one of his strategies 2 and 3 with equal probability.

TABLE 1.

Competitive

		II: 1	2	3	4
I	1	1	9	16	11
	2	20	2	15	12
	3	7	17	5	13
	5	10	6	18	3
	4	19	4	8	14

		I: 1	2	3	4	5
II	1	20	1	14	11	2
	2	12	19	4	15	17
	3	5	6	16	3	13
	4	10	9	8	18	7

TABLE 2.

Generated median competitive

		II: 1	2	3	4
I	1	5	9	16	11
	2	20	3	15	10
	3	7	19	1	13
	4	12	6	18	2
	5	17	4	8	14

		I: 1	2	3	4	5
II	1	19	5	14	11	6
	2	13	20	3	16	18
	3	1	2	15	4	12
	4	10	9	8	17	7

The game of Table 4 is median competitive but is not generated by any competitive game. First, consider markings in the payoff matrix for player I of the outcomes where his payoff is at least $P_I = 13$ and also the payoff to player II is at most $P'_{II} = 8$. An outcome of this marked set can be assured with probability at least 1/2 and, as before, a median optimum strategy for player I is to randomly select one of his strategies 2 and 3 with equal probability. Second, consider markings in the payoff matrix for player II of the outcomes where his payoff is at least $P_{II} = 14$ and also the payoff to player I is at most $P'_I = 7$. An outcome of this marked set can be assured with probability at least 1/2 and, again, a median optimum strategy for player II is to randomly choose one of his strategies 1 and 2 with equal probability. Finally the game is not OPMC generated from a competitive game for player I or player II. This follows from occurrence of the payoffs (18, 10), (3, 3), (11, 18), which could not be obtained through generation from a competitive game when $P_I = 13$, $P_{II} = 14$, $P'_I = 7$, $P'_{II} = 8$.

TABLE 3.

Generated OPMC for player I (not OPMC for player II)

		II			
		1	2	3	4
I	1	11	5	16	9
	2	20	3	15	10
	3	7	19	1	13
	4	12	6	18	2
	5	17	4	8	14

		I				
		1	2	3	4	5
II	1	19	5	14	11	6
	2	13	20	3	16	18
	3	1	2	15	4	12
	4	10	9	8	17	7

TABLE 4.

Median competitive, not generated

		II			
		1	2	3	4
I	1	1	9	16	11
	2	20	2	15	12
	3	7	17	5	13
	4	10	6	3	18
	5	19	4	8	14

		I				
		1	2	3	4	5
II	1	20	1	14	11	2
	2	12	19	4	15	17
	3	5	6	16	3	13
	4	18	9	8	10	7

Next, consider the game of Table 5. For player II, this game is OPMC but not generated by a competitive game. The OPMC part for player II is verified by marking in his matrix the positions of outcomes where his payoff is at least $P_{II} = 14$ and also the payoff to player I is at most $P'_{I} = 7$. An outcome of this marked set can be assured with probability at least 1/2, and, as before, a median optimum strategy for player II is to randomly select one of his strategies 1 and 2 with equal probability. The game is not OPMC generated from a competitive game for player II, as is seen from occurrence of the outcomes (3, 6) and (8, 18). Now consider player I. This game is not OPMC in any sense for player I. Examination shows that $P_{I} = 13$ and $P'_{II} = 7$. In the matrix for player I, let the outcomes be marked which are such that the payoff to player I is at least 13 and also the payoff to player II is at most 8. An outcome of this marked set cannot be assured with probability at least 1/2 by player I.

TABLE 5.
OPMC for player II, not generated (not OPMC for player I)

II

		1	2	3	4
	1	1	9	16	11
	2	20	2	15	12
I	3	7	17	5	13
	4	10	6	18	8
	5	19	4	3	14

I

		1	2	3	4	5
	1	20	1	14	11	2
II	2	12	19	4	15	17
	3	5	13	16	3	6
	4	10	9	8	18	7

TABLE 6.
Not OPMSC for either player

II

		1	2	3	4
	1	16	9	1	11
	2	20	2	15	12
I	3	7	17	5	13
	4	10	8	18	6
	5	19	4	3	14

I

		1	2	3	4	5
	1	20	1	14	11	2
	2	12	19	4	15	17
II	3	5	13	16	3	6
	4	10	9	8	18	7

Finally, consider the game of Table 6. This game is not OPMC, in any sense, for either player. First, consider player I. Examination shows that $P_{I} = 13$ and $P'_{II} = 7$. Let the outcomes where the payoff to player I is at least 13 and also the payoff to player II is at most 7 be marked in the payoff matrix for player I. An outcome of this marked set cannot be assured with probability at least 1/2 by player I. Likewise, let a similar marking be done for player II, where $P'_{II} = 8$ and $P_{II} = 14$. Player II cannot assure an outcome of the marked set with probability at least 1/2.

Appendix.

Considered first is evaluation of P_I, P_{II} and determination of median optimum strategies for the case of players acting protectively. This is followed by an outline of a method to evaluate P'_I, P'_{II} and to determine median optimum strategies for the case of players acting vindictively. Finally, some very easily applied methods that often can be used are presented. These methods are also usable for determining whether a game is OPMC for a player and frequently yield a median optimum strategy when the game is OPMC. The easily applied methods are applicable for all the examples that are considered. The results of this Appendix are implied by the material of ref. 3.

For player I (II) acting protectively, first mark the position(s) in his matrix of the largest payoff value. Then also mark the position(s) of the next to largest payoff value. Continue this marking, according to decreasing payoff value, until the first time that player I (II) can assure a marked value with probability at least 1/2. Then P_I (P_{II}) is the payoff value associated with the last of the markings.

A general method for determining when a marked value can be assured with probability at least 1/2 is obtained by a special use of zero-sum expected-value game theory. Let a modified payoff matrix for player I (II) be determined by replacing every marked payoff by unity and every unmarked payoff by zero. Player I (II) can assure a marked payoff with probability at least 1/2 if and only if the value of this game, to player I (II), is at least 1/2. A protective median optimum strategy for player I (II) is obtained as an optimum strategy for him in the solution of the zero-sum game the first time that the game value is at least 1/2.

Another method, that is much more easily applied, is often usable. Let the marking, according to decreasing payoff value, be continued until the first time that marks in all columns can be obtained from two or fewer rows. Now examine the unmarked positions and suppose that « unmarks » in all rows can be obtained from two or fewer columns. Then, for player I (II), the value of P_I (P_{II}) is the payoff value associated with the last of the markings. If a fully marked row occurs, use of this row provides a protective median optimum strategy. Otherwise, consider any two rows that together have marks in all columns. Random selection of one of these rows, with equal probability, furnishes a protective median optimum strategy.

For player I (II) acting vindictively, first mark the position(s) in the matrix for player II (I) of the smallest payoff value. Then also mark the position(s) of the next to smallest payoff value. Continue this marking, according to increasing payoff value, until the first time that player I (II) can assure a marked value with probability at least 1/2. Then P'_{II} (P'_{I}) is the payoff value associated with the last of the markings.

A general method similar to that for the protective case can be used to determine when a marked value in the matrix for player II (I) can be assured by player I (II) with probability at least 1/2. A modified payoff matrix for player II (I) is determined by replacing every marked payoff by zero and every unmarked payoff by unity. Player I (II) can assure a marked payoff with probability at least 1/2 if and only if the value of this game, to player II (I), is at most 1/2.

Another more easily applied method is frequently usable. Let the marking, according to increasing payoff value, be continued until the first time that marks in all rows can be obtained from two or fewer columns. Examine the unmarked positions and suppose that « unmarks » in all columns can be obtained from two or fewer rows. Then, in the matrix for player II (I), the value of P'_{II} (P'_{I}) is the payoff value associated with the last of the markings. If a fully marked column occurs in the matrix for player II (I), vindictive player I (II) can use this column as a median optimum strategy. Otherwise, consider any two columns that together have marks in all rows. Random selection of one of these two columns, with equal probability, provides a vindictive median optimum strategy.

Finally, consider an easily applied method of determining whether a game is OPMC for a player and, if so, of determining a median optimum strategy. This method is not generally applicable but often is usable. It is simlar to the easily applied methods stated for protective and for vindictive players.

For player I (II) considered, mark the positions in this matrix that correspond to the outcomes of set I (II). The game is OPMC for this player if the marking is such that marks in all columns can be obtained from two or fewer rows. If one row is fully marked, this row provides a median optimum strategy for the player. Otherwise, for the game OPMC to the player, consider the unmarked positions. Suppose that « unmarks » in all rows can be obtained from two or fewer columns. Then, for any two rows that have marks in all columns, a random selection of one of these rows, with probability 1/2 for each row, provides a median optimum strategy.

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