RATIONAL DETERMINATION OF THE MARKETING EXPENDITURE

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SUMMARY. — A mathematical theory is developed for the marketing expenditure, starting from a differential equation, in which the decrease of the marketing elasticity with increasing marketing expenditure is expressed. The integration yield s-shaped curves, and introduces a constant c, *the marketing resistivity,* the numerical value of which is to be determined empirically. This result is perfectly suited to account for the empirical features of marketing, where a characteristic difference exists between various products (cars : low marketing resistivity, cosmetics : high marketing resistivity).

The optimal marketing effort is calculated and yields very reasonable results.

Finally, another type of marketing is discussed, the type in which the marketing expenditure is included in the price ; the results for optimal gain are discussed.

§ 1. The subject of our paper is extensively treated in many handbooks. We refer to Kotler's Marketing Management, p. 272-287, and especially figures 12-2 a and b, and figure 12-5. However, it is easily seen that the given formulae do not represent the drawn curves; the curves have $y = 0$ at $x = 0$; of all the formulae of figures 12(a), 12(b), only the formula $y = ax$ satisfies this requirement. Further, the formulae and curves of figure 12-5 seem acceptable; but if the curves had been prolonged further, they would, according to the proposed formulae, go down again : Decreasing sales with increasing marketing expenditure, which is not acceptable. Therefore this note may be of some interest, the more so because marketing expenditure runs into the billions of dollars.

§ 2. The simplest assumption which can be made in this field is

$$\frac{dS}{S} = \alpha \frac{dM}{M} \tag{1}$$

where dS/S is the fractional increase of sales, dM/M the fractional increase of marketing expenditures, and where α is a constant.

For this case α coincides with e_m, the marketing expenditure elasticity. Although we will have to modify this equations later, let us first write down its integration

$$\ln S = \alpha \ln M + \ln S_l \; ; \qquad S = S_l M^{\alpha} \tag{2}$$

Three types of S (M) curves are possible :

(a) $\alpha < 1$ curve concave towards M-axis; slope at origin infinite

(b) $\alpha = 1$ curve linear; slope finite

(c) $\alpha > 1$ curce convex towards M-axis; slope at origin vanishing.

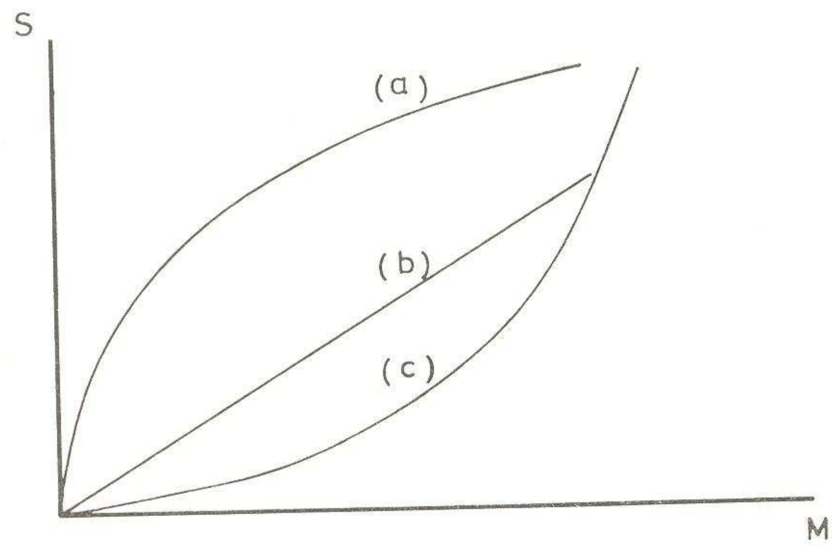

Fig. 1. — S(M) for $\alpha \begin{smallmatrix} < \\ = \\ > \end{smallmatrix} 1$.

Our three curves have one thing in common : for $M = \infty$, S becomes infinite. Since this is clearly impossible, we correct equation (1) by adding an appropriate term. Before doing this, we point out that Kotler's curves (fig. 12-2(a) and fig. 12-5) are of our type (a), and that the lower part of his curve 12-2(b) is of our type (c). In this way we obtain a considerable simplification and clarification.

§ 3. As our next step we'll modify our fundamental differential-equation (1) in such a way that the sales will show a limit S_l for high marketing expenditure. This can be done simply by writing

$$\frac{dS}{S} = \alpha \left(1 - \frac{S}{S_l}\right) \frac{dM}{M} \tag{3a}$$

or, what is equivalent

$$\frac{dS}{dM} = \alpha \left(1 - \frac{S}{S_l}\right) \frac{S}{M} \tag{3b}$$

In fact, we see that for $S = S_l$, $dS/dM = 0$, hence no increase of sales is obtained; while for $S/S_l \ll 1$, we are carried back to equation (1).

We may remark that the elasticity of the marketing expenditure is given by

$$e_m = \frac{dS/S}{dM/M} = \alpha \left(1 - \frac{S}{S_l}\right) \tag{4}$$

Thus we see that e_m is not a constant, but a quantity varying linearly from its maximum value ($= \alpha$) at $S = 0$ to its lowest value ($= 0$) at $S = S_l$.

It is easily seen that integration of equation (3b) will yield curves of the following type.

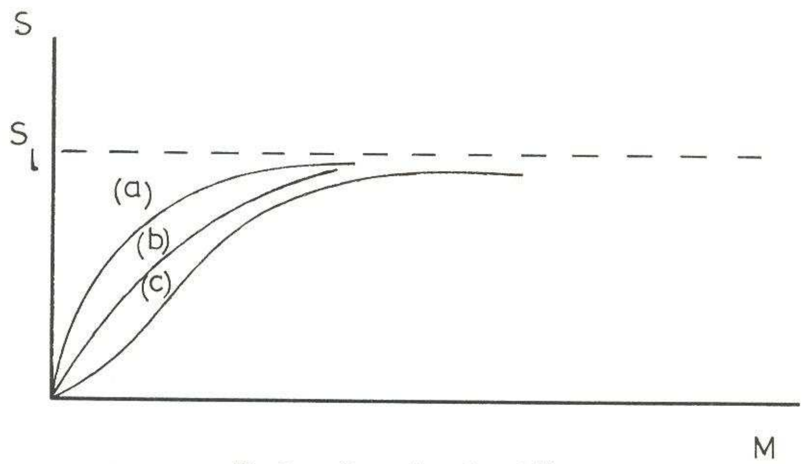

Fig. 2. — Integration of eq. (3b).

It is satisfactory that in this way we have a simple equation giving a curve of the shape of Kotler's [fig. 12-2(b)].

Curves of type (c) have an inflexion point (*i*) and a contact point (*c*) with the tangent from 0. It is easily proved that

$$s_i = \frac{S_i}{S_l} = \tfrac{1}{2} \left(1 - \frac{1}{\alpha}\right) \tag{5}$$

$$s_c = \frac{S_i}{S_l} = 1 - \frac{1}{\alpha} \tag{6}$$

and therefore

$$S_i = \tfrac{1}{2}\, S_c \qquad \text{and} \qquad s_i = \tfrac{1}{2}\, s_c \tag{7}$$

Let us now integrate our equation (3b). We see from equations (5) and (6) that it is useful to express our quantities S and M using S_l as a unit; we write

$$s = \frac{S}{S_l} \qquad\qquad m = \frac{M}{S_l} \tag{8}$$

Equation (3b) then takes the form

$$\frac{ds}{dm} = \alpha\,(1 - s)\,\frac{s}{m} \qquad \text{or} \qquad \frac{ds}{s\,(1 - s)} = \alpha\,\frac{dm}{m} \tag{9}$$

The quantities s and m measure the sales and the marketing expenditure, no longer in dollars, but as a fraction of the sales limit.

Equation (9) can be written as

$$\frac{ds}{s} + \frac{ds}{1 - s} = \alpha\,\frac{dm}{m} \tag{10}$$

Integration gives

$$\ln s - \ln\,(1 - s) = \alpha\,\ln m + \ln c^{-1} \tag{11}$$

where c is an integration constant.

Equation (11) can be written as

$$\frac{s}{1 - s} = c^{-1}\, m^{\alpha} \tag{12}$$

Or also

$$\frac{1 - s}{s} = \frac{c}{m^{\alpha}}$$

$$\frac{1}{s} - 1 = \frac{c}{m^{\alpha}}$$

$$\frac{1}{s} = 1 + \frac{c}{m^{\alpha}} = \frac{m^{\alpha} + c}{m^{\alpha}}$$

$$s = \frac{m^{\alpha}}{m^{\alpha} + c} \tag{13}$$

With this result, our problem, the dependance of s on m has been solved.

First of all, let us represent our result (13) graphically; this has been done in

fig. 3a : $s(m)$ for $\alpha = 2$ and $c = 0.01$, 0.05, 0.1, 0.2, 0.3 and 0.4,

fig. 3b : $s(m)$ for $\alpha = 3$ and $c = 0.01$, 0.5, 0.1, 0.2, 0.3 and 0.4.

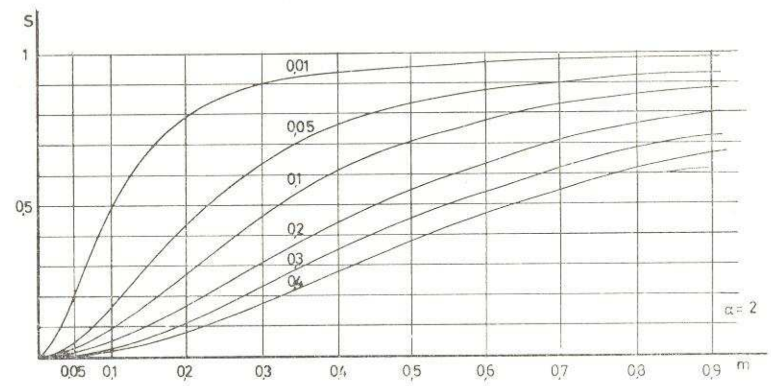

Fig. 3a. — Curves according to eq. (17) for $\alpha = 2$ and $c = 0.01$, 0.05, 0.1, 0.2, 0.3, 0.4. $s_c = 1/2$ $s_i = 1/4$

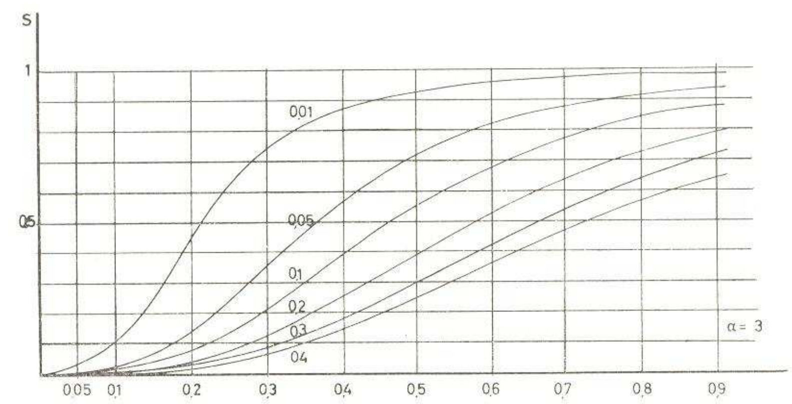

Fig. 3b. — Curves according to eq. (17) for $\alpha = 3$ and $c = 0.01$, 0.05, 0.1, 0.2, 0.3, 0.4. $s_c = 2/3$ $s_i = 1/3$

Our solution contains two constants, α and c; α is a constant occurring already in the differential equation; its value can be determined with the aid of the equations (4) and (8). The constant c, however, is of a quite different character; it has a value which varies in a characteristic way from product to product; it appears from the curves of figures 3a and 3b that the constant c might be called the *marketing resistivity* of the product, for the curves with a high value of c (great marketing resistivity) require a high marketing expenditure in order to attain a reasonable value of s; for curves of low c this expenditure is much lower. This follows also from equation (13) : c is the value of m^{α}, where $s = \frac{1}{2}$; this value will be high for products with high marketing resistivity.

It will be recognized at once that the empirical features of marketing require this conception of marketing resistivity; it is satisfactory that our theory produces this feature without extra assumptions.

To show this explicitly, let us consider how these results must be applied in a particular case. With

$$\alpha = \frac{e_m}{1 - s} \tag{14}$$

we see that a producer has to determine e_m experimentally, and that he must estimate his value of s; then he can calculate α from (14).

Further, if equation (12) is written as

$$c = m^{\alpha} \frac{1 - s}{s} \tag{12}$$

a producer can calculate the value of c for his product from his actual values of m, α and s, and thus give his marketing a scientific basis.

Let us now consider a number of industries, for which we have estimated the values of m from the literature, and for which we have taken $\alpha = 2$, and $s = 0.8$, assuming that this is an acceptable value for a producer who is in business since a long time together with other competitors.

Thus we find for:	m	c
a. The Car Industry	0.08	0.0016
b. General Consumption Articles	0.18	0.008
c. Soap and Washing Preparations	0.32	0.025
d. Cosmetics	0.41	0.04

We now find the following $s\,(m)$ curves.

The quantity c varies strongly from product to product, from 0.0016 for cars to 0.04 for cosmetic articles.

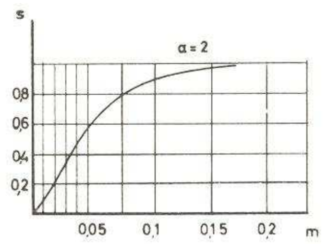

Fig. 4a. — Cars.
$c = 0.0016$.

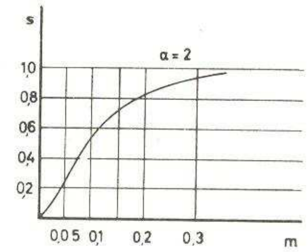

Fig. 4b. — General Consumption Articles. $c = 0.008$.

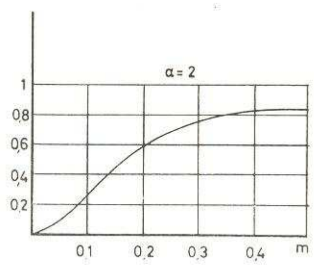

Fig. 4c. — Soap and Washing Prep.
$c = 0.025$.

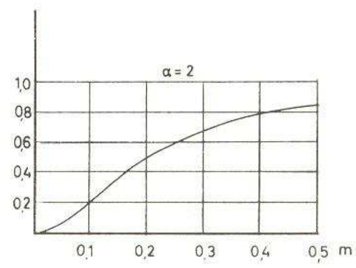

Fig. 4d. — Cosmetics.
$c = 0.04$.

We can read directly from equation (13) that $c = m^{a}$, when m is the (fractional) marketing expenditure necessary to attain 50% of the maximal sales attainable. This is easily verified in all curves of figure 4.

We think that the theory presented here can be characterized in the following way :

In our economical community marketing has developed itself in an autonomic way; thus it has arrived at its present shape, from which one of the most interesting features is the big difference in relative marketing expenditure, when we compare products as sewing machines, or cars, on the one hand, and cosmetics on the other; well then, our theory provides quite naturally the necssary scheme, in which any product finds its appropriate place, by means of the special value of the integration constant c, the marketing resistivity, for each product.

§ 5. We now come to the problem to determine the *optimal marketing expenditure*. According to Kotler (p. 272) one might be inclined to indicate the point of inflection, where

$$s_i = \tfrac{1}{2}\left(1 - \frac{1}{\alpha}\right) \tag{5}$$

but he argues further that this criterion does not make sense. Let us therefore answer our question in a mathematical way.

Let G be the gain, S the sales, C the cost, O the overhead, M the marketing expenditure, and P the material production cost, then we have

$$G = S - C = S - [O + M + P] = S\left(1 - \frac{P}{S}\right) - O - M$$

$$= S(1 - \beta) - O - M \tag{15}$$

where
$$\beta = P/S \tag{16}$$

the material production cost as a fraction of the sales value, a quantity well-known to the producer.

We now have, for the maximum of the G (M) curve

$$\frac{dG}{dM} = (1 - \beta)\frac{dS}{dM} - 1 = 0 \tag{17}$$

$$\frac{dS}{dM} = \frac{1}{1 - \beta} \tag{18}$$

or also

$$\frac{ds}{dm} = \frac{1}{1 - \beta} \tag{19}$$

Now, according to equation (13)

$$s = \frac{m^\alpha}{m^\alpha + c} \tag{13}$$

Hence

$$\frac{ds}{dm} = \frac{(m^\alpha + c)\,\alpha\, m^{\alpha-1} - m^\alpha\,\alpha\, m^{\alpha-1}}{(m^\alpha + c)^2} = \frac{1}{1 - \beta}$$

$$\frac{c\,\alpha\, m^{\alpha-1}}{(m^\alpha + c)^2} = \frac{1}{1 - \beta}$$

$$(m_0^\alpha + c)^2 - (1 - \beta)\,c\,\alpha\, m_0^{\alpha-1} = 0 \tag{20}$$

where m_0 indicates the optimal marketing expenditure.

We solve this equation by writing

$$m_0^{\alpha} + c = \sqrt{(1 - \beta)\; c\; \alpha\; m_0^{\alpha-1}} \qquad (21)$$

and plotting left hand side and right hand side as a function of m; the intersection point Q indicates the value of m_0.

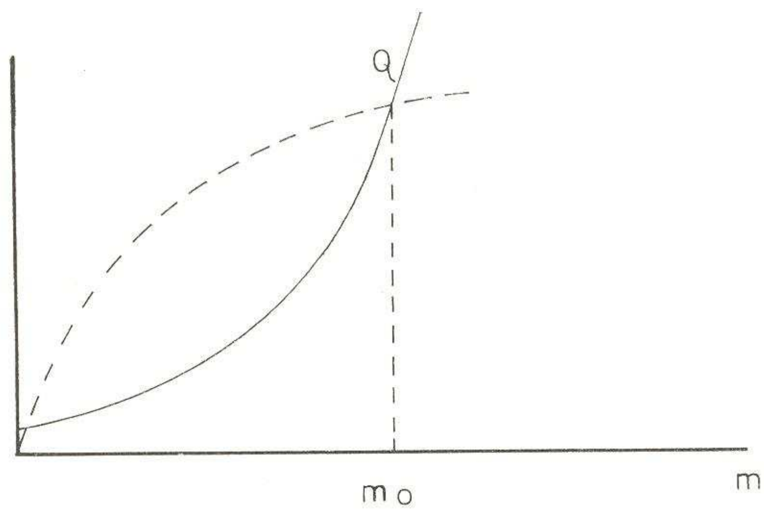

Fig. 5. — How to solve the eq. (21).

Often the contribution of c is not important in m_0^{α}; then, for a first approximation, we may there put $c = 0$; for the case $\alpha = 2$ we then find

$$m_0^3 = (1 - \beta)\; c\; \alpha \qquad (22)$$

For the car industry, with, say, $\beta = 4/5$, $1 - \beta = 1/5$, $c = 0.0016$, we find

$$m_0^3 = 0.000640; \qquad m_0 = 0.086$$

which is quite a reasonable answer (cf. fig. 4a) for it corresponds with $s = 0.82$.

We may point out, perhaps, that m_0 is very far away from m_i (Kotler, p. 272), which is in our case equal to 0.023.

[For the inflexion-point we have, according to (5)

$$s_i = \tfrac{1}{2}\,(1 - \frac{1}{\alpha}) = 0.25; \qquad \frac{1}{s_i} = 4$$

Further, acording to equation (12), with $\alpha = 2$, we have

$$m_1^2 = \frac{c}{(1/s_1) - 1} = \frac{0.0016}{3} + 0.00053; \qquad m_1 = 0.023].$$

It is easily seen that our theory works also satisfactorily at the other end of the range (fig. 4d). If we put here $\beta = 1/5$, $1 - \beta = 4/5 = 0.8$, $\alpha = 2$, $c = 0.04$, we find

$$m_0^3 = 0.060; \qquad m_0 = 0.4$$

which is again quite reasonable, for it corresponds with $s = 0.80$; and higher values of s would require much heavier marketing expenditure.

§ 6. It is, perhaps, useful to point out that we have taken the producer's sales price as fixed; this means that the gain increases because the number of units sold increases when the marketing effort is intensified.

In another industry there might be no need to keep the sales price fixed, but to let it absorb the marketing expenditure partly or wholly.

Only in this last case the problem is completely defined. Now, in the derivation of equation (3a), the unit price p was assumed to be a constant; if this is no longer true, we write, in stead of (3a)

$$\frac{dN}{N} = e_m \frac{dM}{M} - e_p \frac{dp}{p} \tag{23}$$

Here we have introduced N, the number of units sold; e_m is the marketing elasticity; and we had to include in the right hand side of equation (23) a term $e_p\ dp/p$, expressing the relative decrease in the number of units sold, caused by a relative increase in price; e_m and e_p may be functions of N.

Now we have

$$p = p_p + g + m \tag{24}$$

The price is composed of production price, gain, and marketing expenditure, all per unit.

The total gain is

$$G = Ng \tag{25}$$

Using

$$M = Nm; \quad \ln M = \ln N + \ln m; \quad \frac{dM}{M} = \frac{dN}{N} + \frac{dm}{m} \tag{26}$$

we obtain, from equations (23) and (26)

$$\frac{dN}{N} = e_m \left(\frac{dN}{N} + \frac{dm}{m}\right) - e_p \frac{dp}{p} \tag{27}$$

We have further, from (24)

$$dp = dm \tag{28}$$

Hence

$$(1 - e_m) \frac{dN}{N} = \left(\frac{e_m}{m} - \frac{e_p}{p}\right) dm \tag{29}$$

In this case the gain is optimal when N is optimal, when $dN = 0$, or $dN/N = 0$, hence when

$$\frac{e_m}{m_0} = \frac{e_p}{p} \tag{30}$$

Or when

$$m_0 = \frac{e_m}{e_p} p = \frac{e_m}{e_p} (p_p + g + m_0) \tag{31}$$

$$m_0 \left(1 - \frac{e_m}{e_p}\right) = \frac{e_m}{e_p} (p_p + g)$$

$$m_0 = \frac{e_m/e_p}{1 - e_m/e_p} (p_p + g) \tag{32}$$

Discussion.

It follows from equation (23) that our e_p is defined as a *positive* quantity.

$e_p > e_m$ Equation (32) gives us the optimal value of m

$e_p = e_m$ There is no optimal value of m; the producer, who includes his marketing expenditure in his price can go on to do so, untill e_m will become smaller than e_p

$e_p < e_m$ .As sub $e_p = e_m$.