EXCEEDANCE AREAS FOR SECOND ORDER STOCHASTIC PROCESSES

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Summary.

Expressions are derived for the mean value and variance of the areas of exceedance of functional bounds for second order stochastic process $X(t)$ in the interval $(0, T)$ and applied specially to normal stochastic processes Finally, bounds for exceedance planning are obtained.

1. Introduction.

In the study of the behaviour of the stochastic process associated with the deterioration or aging of many engineering systems, one of the performance indexes used is the random time that the stochastic process exceeds some previous bounds.

A close scruting of the problem suggests that in many situations, as deterioration studies, the random amount of time spent outside the bounds is not very important provided that the amount of exceedance is small and, on contrary, if the amount of exceedance is large the deterioration of the system may be very large even if the time of exceedance is short.

The deterioration may be considered, in simple cases, simultaneously proportional to the intensity and the duration of the exceedance, that is, to the area in which the excursion of the stochastic process exceeds the bounds. We can even, more generally, introduce a response function or deterioration intensity, the area corresponding to the proportional response function.

In many situations we can, only, be interested in upper or lower exceedances (that is, the exceedance of the upper or lower bounds); we will also consider, for the exceedance of both bounds, the case in which there is no compensation by the exceedance of the bounds as well as the case in which the exceedances, having different signs, may compensate the effects.

The bounds are considered as levels of risky situations to be avoided; we will assume that they may be variable with time.

A study of this problem for a normal stationary process was made by Leadbetter (1963).

2. Basic Results.

Let $X(t)$ $(0 \leqslant t \leqslant \infty)$ be a second order stochastic process with an almost surely continuous sample function, mean value function $\mu(t)$, variance function $\sigma^2(t)$ and correlation function $\rho(t, s)$ [the correlation between $X(t)$ and $X(s)$].

Let $\underline{b}(t)$ and $\overline{b}(t)$ $[-\infty \leqslant \underline{b}(t) \leqslant \overline{b}(t) \leqslant +\infty]$ be the bounds in whose exceedance we are interested. In practical problems we will, in general, use bounds such that

$$\underline{b}(t) \leqslant \mu(t) \leqslant \overline{b}(t), \tag{1}$$

and, even more, symmetric bounds such that

$$\underline{b}(t) + \overline{b}(t) = 2\,\mu(t). \tag{2}$$

Associated with the stochastic process we can define two exceedance stochastic processes (upper and lower) as

$$\begin{aligned} \overline{Y}(t) &= X(t) - \overline{b}(t) && \text{if } X(t) > \overline{b}(t) \\ &= 0 && \text{if } X(t) \leqslant \overline{b}(t) \end{aligned} \tag{3}$$

and

$$\begin{aligned} \underline{Y}(t) &= \underline{b}(t) - X(t) && \text{if } X(t) < \underline{b}(t) \\ &= 0 && \text{if } X(t) \geqslant \underline{b}(t). \end{aligned} \tag{4}$$

The total exceedance $Y(t)$ and net exceedance $Y_0(t)$ stochastic processes are evidently given by the relations

$$\begin{aligned} Y(t) &= \underline{Y}(t) + \overline{Y}(t) = |Y_0(t)| \\ Y_0(t) &= \overline{Y}(t) - \underline{Y}(t) \end{aligned} \tag{5}$$

$\underline{Y}(t)$, $\overline{Y}(t)$ and $Y(t)$ are always non-negative.

The random areas of exceedance are then, for the interval (0, T),

upper exceedance area $\overline{A}(T) = \int_0^T \overline{Y}(t)\, dt$

lower exceedance area $\underline{A}(T) = \int_0^T \underline{Y}(t)\, dt$

$$\text{total exceedance area} \quad A(T) = \int_0^T Y(t)\, dt = \overline{A}(T) + \underline{A}(T)$$

$$\text{net exceedance area} \quad A_0(T) = \int_0^T Y_0(t)\, dt = \overline{A}(T) - \underline{A}(T)\,.$$

The two random variables $\overline{A}(T)$ and $\underline{A}(T)$ are evidently the basic ones and, in the sequel we will deal only with them. Let, then, denote by $\overline{m}(T)$, $\underline{m}(T)$, $\overline{s}^2(T)$, $\underline{s}^2(T)$ and $r(T)$ the mean values, variances and correlation coefficient of $\overline{A}(T)$ and $\underline{A}(T)$; the analogous values for the other areas are immediately deduced from previous formulae.

We will compute, now, the general expressions of this quantities. Let $F(x;t)$ and $G(x, y; t, s)$ be the distribution functions of $X(t)$ and of the pair $[X(t), X(s)]$.

The mean values have then the expressions (m denoting the mean value operator)

$$\overline{m}(T) = \int_0^T m\, [\overline{Y}(t)]\, dt = \int_0^T \left\{ \int_{b(t)}^{+\infty} [1 - F(x;t)]\, dx \right\} dt \tag{6}$$

and similarly

$$\underline{m}(T) = \int_0^T \left[\int_{-\infty}^{b(t)} F(x;t)\, dx \right] dt \tag{7}$$

For the computation of the variances and covariances we have

$$\overline{s}^2(T) = \int_0^T \int_0^T \operatorname{cov}\, [\overline{Y}(t), \overline{Y}(s)]\, dt\, ds$$

with

$$\operatorname{cov}\, [\overline{Y}(t), \overline{Y}(s)] = \int_{b(t)}^{+\infty} \int_{b(s)}^{+\infty} [G(x, y; t, s) - F(x;t)\, F(y;s)]\, dx\, dy \tag{8}$$

$$\underline{s}^2(T) = \int_0^T \int_0^T \operatorname{cov}\, [\underline{Y}(t), \underline{Y}(s)]\, dt\, ds$$

with

$$\operatorname{cov}\, [\underline{Y}(t), \underline{Y}(s)] = \int_{-\infty}^{b(t)} \int_{-\infty}^{b(s)} [G(x, y; t, s) - F(x;t)\, F(y;s)]\, dx\, dy \tag{9}$$

and

$$r(T)\, \overline{s}(T)\, \underline{s}(T) = \int_0^T \int_0^T \operatorname{cov}\, [\overline{Y}(t), \underline{Y}(s)]\, dt\, ds$$

with

$$\operatorname{cov}[\overline{Y}(t), \underline{Y}(s)] = \int_{\overline{b}(t)}^{+\infty} \int_{-\infty}^{\underline{b}(s)} [F(x;t)\, F(y;s) - G(x,y;t,s)]\, dx\, dy \quad (10)$$

In the cases of symmetry we have :

$$\overline{m}(T) = \underline{m}(T)$$
$$\overline{s}(T) = \underline{s}(T)\,. \quad (11)$$

3. Some Special Processes.

Let us now suppose that $X(t) - \mu(t) / \sigma(t)$ is stationary of first order, that is,

$$F(x;t) = F_0[x - \mu(t) / \sigma(t)] \quad (12)$$

and let G_0 be such

$$G(x,y;t,s) = G_0[x - \mu(t) / \sigma(t),\ y - \mu(s) / \sigma(s);\ t, s] \quad (13)$$

Supposing, also, that the reduced bounds are constants $\underline{\beta}$ and $\overline{\beta}$ we have

$$\overline{m}(T) = \int_{\overline{\beta}}^{+\infty} [1 - F_0(x)]\, dx \,.\, \int_0^T \sigma(t)\, dt$$

$$\underline{m}(T) = \int_{-\infty}^{\underline{\beta}} F_0(x)]\, dx \,.\, \int_0^T \sigma(t)\, dt$$

$$\overline{s}^2(T) = \int_0^T \int_0^T \sigma(t)\, \sigma(s) \left\{ \int_{\overline{\beta}}^{+\infty} \int_{\overline{\beta}}^{+\infty} [G_0(x,y;t,s) - F_0(x)\, F_0(y)]\, dx\, dy \right\} dt\, ds$$

$$\underline{s}^2(T) = \int_0^T \int_0^T \sigma(t)\, \sigma(s) \left\{ \int_{-\infty}^{\underline{\beta}} \int_{-\infty}^{\underline{\beta}} [G_0(x,y;t,s) - F_0(x)\, F_0(y)]\, dx\, dy \right\} dt\, ds \quad (14)$$

and

$$r(T)\, \overline{s}(T)\, \underline{s}(T)$$

$$= \int_0^T \int_0^T \sigma(t)\, \sigma(s) \left\{ \int_{\overline{\beta}}^{+\infty} \int_{-\infty}^{\underline{\beta}} [F_0(x)\, F_0(y) - G_0(x,y;t,s)]\, dx\, dy \right\} dt\, ds \quad (15)$$

Let us, finally, consider a normal stochastic process. It satisfies previous considerations with $F_0(x) = \Phi(x)$ the standard normal distribution function and $G_0(x, y; t, s) = \Phi_\rho(x, y)$ the binormal distribution function with standard margins and correlation coefficient $\rho = \rho(t, s)$; we will also suppose the symmetry of the reduced bounds $\overline{\beta} = c > 0$, $\underline{\beta} = -c$.

Applying the formulae obtained we have, owing to the development

$$\Phi_{\rho}(x, y) = \sum_{j=0}^{\infty} \frac{\rho^j}{j!} \Phi^{(j)}(x) \Phi^{(j)}(y) \tag{16}$$

given in Cramer (1946),

$$\overline{m}(T) = \underline{m}(T) = \int_0^T \sigma(t)\, dt \int_c^{+\infty} [1 - \Phi(x)]\, dx \tag{17}$$

$$\overline{s}^2(T) = \underline{s}^2(T) =$$

$$= \int_0^T \int_0^T \sigma(t)\, \sigma(s) \left\{ \int_c^{+\infty} \int_c^{+\infty} [\Phi_{\rho}(x, y) - \Phi(x)\, \Phi(y)]\, dx\, dy \right\} dt\, ds$$

$$= \int_0^T \int_0^T \sigma(t)\, \sigma(s) \left\{ \sum_{j=1}^{\infty} \frac{\rho^j}{j!} [Q^{(j-1)}(c)]^2 \right\} dt\, ds \tag{18}$$

where $Q(x) = 1 - \Phi(x)$ (using integration by parts) and, also,

$$r(T)\, \overline{s}(T)\, \underline{s}(T) = \int_0^T \int_0^T \sigma(t)\, \sigma(s) \left\{ \sum_{j=1}^{\infty} \frac{(-\rho)^j}{j!} [Q^{(j-1)}(c)]^2 \right\} dt\, ds \tag{19}$$

Let us apply, now, those considerations to the normal stationary and brownian motion stochastic processes.

In the first case we have $\rho(t, s) = \rho(|t - s|)$ and $\sigma(t) = 1$ (say) so that

$$\overline{m}(T) = \underline{m}(T) = T \,.\, \int_c^{+\infty} [1 - \Phi(x)]\, dx \tag{20}$$

$$\overline{s}^2(T) = \underline{s}^2(T) = 2 \sum_{j=1}^{\infty} \left(\frac{Q^{(j-1)}(c)}{j!}\right)^2 \int_0^T (T - \tau)\, \rho^j(\tau)\, d\tau \tag{21}$$

and

$$r(T)\, \overline{s}(T)\, \underline{s}(T) = 2 \sum_{j=1}^{\infty} \left(\frac{Q^{(j-1)}(c)}{j!}\right)^2 \int_0^T (T - \tau)\, \rho^j(\tau)\, d\tau \tag{22}$$

which are contained in Leadbetter (1963) results.

For the brownian motion process as we have $\mu(t) = 0$, $\sigma(t) = \sqrt{t}$ (say) and $\rho(t, s) = \min(t, s) / \sqrt{ts}$ we obtain, after simple computations,

$$\overline{m}(T) = \underline{m}(T) = \frac{2}{3} T^{3/2} \int_c^{+\infty} [1 - \Phi(x)]\, dx \tag{23}$$

$$\bar{s}^2(T) = \underline{s}^2(T) = \frac{4\,T^3}{3} \sum_{j=1}^{\infty} \frac{[Q^{(j-1)}(c)]^2}{j!\,(j+3)} \tag{24}$$

and

$$r(T)\,\bar{s}^2(T) = \frac{4\,T^3}{3} \sum_{j=1}^{\infty} (-1)^j \frac{[Q^{(j-1)}(c)]^2}{j!\,(j+3)} \tag{25}$$

from which follows that

$$r(T) = \sum_{j=1}^{\infty} (-1)^j \frac{[Q^{(j-1)}(c)]^2}{j!\,(j+3)} \Big/ \sum_{j=1}^{\infty} \frac{[Q^{(j-1)}(c)]^2}{j!\,(j+3)} \tag{26}$$

independent of T, a result that could be expected.

4. Exceedance Planning.

Let us deal now with the problem of obtaining information for planning, based on previous sample results. For simplicity of exposition we will deal only with the essentially positive total area, the signed net area and the upper and lower areas being dealt with similarly.

Let $A_1(T), \ldots, A_n(T)$ be a sample of n random total areas for n excursions of the stochastic process $X(t)$ and let $K(a)$ be the distribution function of the (random) total area $A(T)$.

For the next observation of an excursion of the stochastic process we have

$$\text{Prob}\,\{A_{n+1}(T) \leqslant \max\,[A_1(T), \ldots, A_n(T)]\} = \int_0^{+\infty} K(a)\, dK^n(a) = n/n+1\,;$$

if we have 19 excursions of the stochastic process, for instance, the probability that in the 20th excursion the maximum of the observed total areas will be exceeded is 5%.

We could suggest the use of a safety factor ν (> 1) — see Freudenthal (1963) — and search what is the value of

$$\inf \text{Prob}\,\{A_{n+1}(T) \leqslant \nu \max\,([A_1(T), \ldots, A_n(T)]\} \geqslant n/n+1\,.$$

It is sufficient to take the family of distribution functions

$$\begin{aligned} K_\tau(a) &= 0 && \text{if} \quad a < 0 \\ &= a^\tau && \text{if} \quad 0 \leqslant a \leqslant 1 \qquad (\tau > 0) \\ &= 1 && \text{if} \quad 1 < a \end{aligned}$$

to obtain

$$\inf_{T} \int_{0}^{+\infty} K_{T}(v\, a)\; d K_{T}^{n}(a) = n/n+1\,;$$

as it stands the lower bound cannot be improved. The same san be said about the more general problem dealt with in the following.

Consider now the question of evaluating the probability that, from a previous sample of n excursions of a stochastic process, in the next m excursions the total area will be between the ith and jth ordered total areas. Its value is evidently given by

$$\frac{n!}{(i-1)\,(j-i-1)\,(n-j)!}$$

$$\iint_{0 \leqslant y \leqslant z \leqslant +\infty} K^{i-1}(y)[\,K(z) - K(y)]^{j-i-1+m}\,[1 - K(z)]^{n-j}\; dK(y)\; dK(z)$$

$$= \frac{n!}{(n+m)!} \cdot \frac{(j-i-1+m)!}{(j-i-1)!} \tag{27}$$

so that for n and m fixed we can search the values of i and j to obtain some level of probability, provided they are compatible; see Tiago de Oliveira (1952).

The use of a safety factor for the average does not seem very promising for exceedance planning, as it can be easily seen.

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