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EXPERIENCE RATING PERTURBED BY A BROWNIAN MOTION

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ABSTRACT

This paper gives a generalization of a risk process under experience rating in the sense that a Brownian motion is added to the classical model. When the aggregation of claims up to time t, is a diffusion or a compound Poisson process, the probabilities of ruin, both in transient and infinite horizon time, are studied.

1. *INTRODUCTION*

The problem of perturbed experience rating

The principle of experience rating is to adjust premiums continuously (in our paper) on the basis of previous information. Premiums should match the amount of claims and should, at the same time, if possible, take into account the market environment. For example, when the profitability is good, the solvency margin increases to a high level, this stimulates competition and implies new companies drawing up tariff or premium reductions (which suppose that free competition is authorised). Conversely, when the profitability is bad, the insurer should collect more money and consequently increase premiums to face risk exposure. A familiar example is the bonus-malus rating in automobile insurance. For these reasons, we can consider an "experience rating" mathematical model.
Nevertheless, there is a difference betwcen examining premiums in theoretical way and how they actually appear in reality. Actually in practice, the insurer uses "some kind" of experience rating system, which is not based only on risk-theoretical bases but also on other circumstances, let us say, indirect influence factors like :

1) uncertainty on inflation ;
2) Up to date statistics not being available at the time of calculation ;
3) uncertainty due to a lack of precise knowledge about economic activity.
etc.

For example, when industrial and commercial businesses are undergoing a tremendous upswing, this tends to accelerate motor and other traffic, which in turn, tends to increase the number of claims.
On the other hand, during recessions the effects are mainly opposite. So to take into account these indirect influences

we will add, to the "experience rating" model, a perturbation by introducing a Brownian motion for the continuous case considered here (section 3).
In section 4 and 5 we study the case where the aggregate claims up to time t is a Brownian motion with drift and compound Poisson processes respectively.
Moreover, we apply the results of GERBER's paper 1973 [4] to calculate an upper bound for the ruin probability before time t.

Remark

PENTIKAINEN AND RANTALA [9,10] in their studies of the insurance industry in Finland, suggested, in §2.2. Vol.II "Models for premium fluctuation", to perturb the experience rating model with a "white noise" (discrete case) and gave solution to premium calculation for a very simple case.

2. DESCRIPTION OF THE RISK PROCESS

We consider a risk process in which the total premiums received in the time-interval $[0,t]$ is denoted by $P(t)$, and $(S(t),\ t \geq 0)$ represents aggregation of claims up to time t, we assume that the processes $P(t)$ and $S(t)$ are Markovian and defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$.
Finally, let $Z(t)$ be a surplus of a company at time t, $t \geq 0$ and write x for $Z(0)$. We have

$$(1) \qquad Z(t) = x + P(t) - S(t) \quad , \quad t \geq 0.$$

Obviously, $Z(t)$ is one-dimensional Markov process.

3. *EXPERIENCE RATING PERTURBED*

Consider a risk process satisfying (1) except that each element of premium paid is modified by a refund or surcharge according to the stochastic differential equation :

(2) $d\,P(t) = (p - k\,(P(t) - S(t))dt + \sigma dW(t)$

with $P(0) = 0$ a.s.

and where : (i) p is the base premium constant rate

(ii) $(W(t),\ t \geqslant 0)$ is a standard Wiener process independent of $(S(t),\ t \geqslant 0)$

(iii) σ is a positive constant, k being the "experience rating factor" $(0 < k < 1)$.

Equation (2) is a linear stochastic differential equation. From GIHMAN AND SKOROHOD [6] we have the solution

$$(3)\quad P(t) = \exp({}_o\!\int^t - kds)[{}_o\!\int^t \exp(-{}_o\!\int^t - kdu).(p + kS(s))ds + {}_o\!\int^t \exp(-{}_o\!\int^t - kdu)\sigma dW(s)]$$

or equivalently

$$(3')\quad P(t) = e^{-kt}(\frac{p}{k}(e^{kt} - 1) + k\,{}_o\!\int^t e^{ks}S(s)ds + \sigma X(t))$$

where we define $X(t) = {}_o\!\int^t e^{ks}\,dW(s)$.

From the relation (1), it follows that

$$(4)\quad Z(t) = x + \frac{p}{k}(1 - e^{-kt}) + k\,e^{-kt}\,{}_o\!\int^t e^{ks}S(s)\,ds + e^{-kt}\,\sigma X(t) - S(t).$$

In view to characterize and reduce this expression we have the following two propositions.

Proposition 1

$X(t)$ <u>is a gaussian process with zero mean and with covariance</u> :

$$(5)\ \operatorname{cov}(X(s),\ X(t)) = {}_{o}\int^{\min(s,t)} e^{2ku}\, du$$

For the proof see for example ARNOLD [1] chapter 5.
By elementary computation we can write relation (5) as

$$(5)\ \operatorname{cov}(X(s),\ X(t)) = \frac{1}{2k}\,(e^{2k(\min(t,s)} - 1).$$

Let $I(t) = {}_{o}\int^{t} e^{ks}\, d\eta(s)$ where
$(\eta(t),\ t \geqslant 0)$ is a stochastic process defined on $(\Omega, \mathfrak{F}, \mathbb{P})$, and having stationary, independent increments, finite variance with $\eta(0) = 0$ and belonging to $D\,[0,\infty)$, where $D(0,\infty)$ denote the space of functions on $[0,\infty)$ that are right-continuous and have left hand limites.
We have the following result, justifying the integration by parts for the stochastic integral $I(t)$.

Proposition 2*

<u>The process</u> $(I(t),\ t \geqslant 0)$ <u>is well defined, a.s. finite, and every sample path satisfies the following relation</u> :

$$(6)\ I(t) = e^{kt}\eta(t) - k\ {}_{o}\int^{t} e^{ks}\, \eta(s)ds.$$

Furthermore, $I(t)$ is a.s. in $D[0,\infty)$

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* This proposition was pointed out by Harrison in [7].

Proof :
Since e^{kt} is a continuous function of bounded variation, we can apply lemma 1 chapter 3 of [2] for the function $\eta(t)$; then the proposition follows form theorem 2 of DUNFORD and SCHWARTZ [3, p.154].

So, from (6) put $\eta(t) \equiv S(t)$ (when $S(t)$ satisfies the conditions on η) we can rewrite (4) as

$$(7)\quad Z(t) = x + \frac{p}{k}(1 - e^{-kt}) - e^{-kt} \int_0^t e^{ks}\, d\, S(s)$$
$$+ e^{-kt}\, \sigma X(t)$$

4. *THE DIFFUSION PROCESS*

Assume now that $S(t)$ satisfies the differential (stochastic) equation :

$$dS(t) = m dt + \sigma_1\, dW_1(t)$$

where m is a constant and $W_1(t)$ is a standard Wiener process independent of $W(t)$.
Then the relation (7) gives

$$(8)\quad Z(t) = x + \frac{p}{k}(1 - e^{-kt}) - e^{-kt} \int_0^t e^{ks}\, m ds$$
$$+ e^{-kt}(\sigma_1\, X_1(t) + \sigma X(t))$$

where we define, as before,

$$X_1(t) = \int_0^t e^{ks}\, dW_1(s)$$

From proposition 1 $(X_1(t),\ t \geqslant 0)$, is a gaussian process independent of $(X(t),\ t \geqslant 0)$ with zero mean and as covariance function :

$$\operatorname{cov}(X_1(s),\ X_1(t)) = \frac{1}{2k}\, (e^{2k(\min(t,s))} - 1)$$

It is well known that the sum of two independent gaussian processes is a gaussian one. So we can write :

$$(\sigma_1 X_1(t) + \sigma X(t)) = \hat{\sigma}\hat{X}(t)$$

where (i) $(\hat{X}(t),\ t \geqslant 0)$ is a gaussian process with zero mean and having the same covariance function of $(X_1(t),\ t \geqslant 0)$

(ii) $\hat{\sigma}^2 = \sigma_1^2 + \sigma^2$.

So write from relation (8)

$$Z(t) = x + \hat{Z}_1(t)$$

with

$$(9)\ \hat{Z}_1(t) = \frac{p-m}{k}(1 - e^{-kt}) + \hat{\sigma} e^{-kt} \hat{X}(t)$$

It is clear that $(\hat{Z}_1(t),\ t > 0)$ is a gaussian with independent increments.

4.1. An upper bound on the probability of ruin

We are interested in the variable "time of ruin" defined as usual by :

$$T = \inf \{t \geqslant 0\ ;\ Z(t) < 0\}$$

Introduce the usual probabilities of ruin, respectively on finite and infinite horizons :

$$\Psi(x,t) = \mathbb{P}\,[\ T \leqslant t\ /\ Z(0) = x],$$
$$\Psi(x) = \mathbb{P}\,[\ T < \infty\ /\ Z(0) = x].$$

GERBER [4] shows that

$$(10)\ \Psi(x,t) \leqslant \min_r e^{-rx} \max_{0 \leqslant s \leqslant t} \mathbb{E}[e^{-r\,\hat{Z}_1(s)}].$$

Now, as

$$\hat{Z}_1(t) \sim \mathcal{N}(m(t),\ s(t))$$

with

$$m(t) = \frac{\mu}{k}(1 - e^{-kt})$$

$$s^2(t) = \frac{\hat{\sigma}^2}{2k}(1 - e^{-2kt})$$

where $\mu = p - m$,
we can write

$$(11) \quad \mathbb{E}[e^{-r\hat{Z}_1(t)}] = \exp\left[- rm(t) + \frac{1}{2} s^2(t) r^2\right]$$

For fixed t, the exponent in (11) is 0 if $r_1 = r = 0$,

or $r = r_2(t) = \dfrac{4\mu}{\hat{\sigma}^2} \dfrac{1}{1 + e^{-kt}}$

so, for $r > r_2(t)$ it is positive and increasing.
Consequently, the maximum in (10) is 1 if $0 \leqslant r < r_2(t)$

This reduces (10) to :

$$(12) \quad \Psi(x,t) \leqslant \min_{r \geqslant r_2(t)} \exp\{-rx - r\frac{\mu}{k}(1 - e^{-kt}) + \frac{\hat{\sigma}^2}{4k}(1-e^{-2kt})r^2\}$$

We find (by differentiation) that the minimum is assumed by

$$(13) \quad r_{min} = \frac{2}{\hat{\sigma}^2} \frac{kx + \mu(1 - e^{-kt})}{(1-e^{-2kt})}$$

Consequently, we have :

$$(14) \quad \Psi(x,t) \leqslant \exp\{\frac{-1}{\hat{\sigma}^2 k} \frac{[kx + \mu(1 - e^{-kt})]}{(1 - e^{-2kt})} \quad \text{if} \quad \frac{\mu(1 - e^{-kt})}{kx} < 1$$

4.2. The ultimate ruin is certain

In order to calculate $\Psi(x)$, recall that

$$(15)\quad Z(t) = e^{-kt}[x\, e^{kt} + \frac{\mu}{k}(e^{kt} - 1) + \hat{\sigma}\, \hat{X}_t]$$

and define

$$(16)\quad \zeta(t) = x\, e^{kt} + \frac{\mu}{k}(e^{kt} - 1) + \hat{\sigma}\, \hat{X}_t$$

Obviously $T = \inf\{t \geqslant 0\ ;\ \zeta(t) \leqslant 0\}$.
We have

Proposition 3

$Z(t)$ is a diffusion process with a drift $\mu(y) = \mu^* - ky$, where $\mu^* = \mu + kx$ and an infinitesimal variance : $\sigma^2(y) = \hat{\sigma}^2$

It is clear that $Z(t)$ is gaussian and has continuous sample paths with independent increments, the first two moments of this process are :

$$\mathbb{E}\, Z(t) = x + \frac{\mu}{k}(1 - e^{-kt})$$
$$\operatorname{var}(Z(t)) = \frac{\hat{\sigma}^2}{2k}(1 - e^{-2kt})$$

Thus, we can represent $Z(t)$, by what HARRISON [7] called, compounding Brownian motion,

$$(17)\quad Z(t) = x + \frac{\mu}{k}(1 - e^{-kt}) + \frac{\hat{\sigma}}{\sqrt{2k}}\, W(1 - e^{-2kt}) \qquad t \geqslant 0$$

From this, it follows that Z is a strong Markov with stationnary transition probabilities, so it is a diffusion.
An elementary computation show that

$$\mu(y) = \lim_{\Delta t \to 0} \frac{1}{\Delta t}\, \mathbb{E}[Z(t + \Delta t) - Z(t)\ /\ Z(t) = y] = (\mu + kx) - ky.$$

and

$$\sigma^2(y) = \lim_{\Delta t \to 0} \frac{1}{\Delta t}\, \mathbb{E}[(Z(t + \Delta t) - Z(t))^2\ /\ Z(t) = y] = \hat{\sigma}^2$$

Consequence

In fact $Z(t)$ is an Ornstein-Uhlenbeck (O.U.) process. To verifie it, let us recall the classical O.U. denoted by $Z_1(t)$:

$$Z_1(t) = e^{-\alpha t} W_1(e^{2\alpha t})$$
$$= e^{-\alpha t} \int_{-\infty}^{t} \sqrt{2\alpha}\, e^{\alpha s}\, dW_2(s)$$

where W_i $i = 1,2$ are two copies of a Brownian motion and a positif constant.

Obviously, from proposition 3, $Z(t)$ is an O.U. process with $Z(0) = x$ and it is well known that the O.U. process reaches with certainty the exterior of the interval $(0,\infty)$, which implies, for our problem, that the ruin is certain.
Another proof is the following, let us represent

$\zeta(t)$ by the compounding Brownian motion :

$$(19) \quad \zeta(t) = xe^{kt} + \frac{\mu}{k}(e^{+kt} - 1) + \frac{\hat{\sigma}}{\sqrt{2k}} W(e^{2kt} - 1) \qquad t \geqslant 0$$

and let
$$v = e^{2kt} - 1$$

and
$$v^* = e^{2kT} - 1$$

So that v^* is the first $v \geqslant 0$ such that

$$(20) \quad x\sqrt{v+1} + \frac{\mu}{k}(\sqrt{v+1} - 1) + \frac{\hat{\sigma}}{\sqrt{2k}} W(v) = 0 \qquad v \geqslant 0$$

or

$$(21)\quad W(v^*) = -\alpha\sqrt{v^* + 1} + \beta \equiv f(v^*)$$

with

$$\alpha = \frac{\sqrt{2k}}{\hat{\sigma}}\left(x + \frac{\mu}{k}\right)$$

and

$$\beta = \frac{\mu}{k}\,\frac{2k}{\hat{\sigma}}$$

from the fundamental Wald identity in continuous time, applied to $f(v)$, (see for example SHEPP [12]), it follows that $\Psi(X) = 1$ a.s.

5. *THE COMPOUND POISSON PROCESS.*

Let $S(t)$ be a compound Poisson process; we can write :

$$(22)\quad S(t) = \sum_{i=1}^{N(t)} A_i$$

where $\{A_i\}_{i \geqslant 1}$ is a sequence of positive independent, identically distrubuted random variables with a common distribution function $F(.)$, and $\{N(t),\ t \geqslant 0\}$ is a Poisson stochastic process, independent of the $\{A_i\}_{i \geqslant 1}$, having parameter λ.

Moreover, we assume $S(t)$ independent of $(X(t),\ t \geqslant 0)$, defined in section 3. In the context of classical risk theory :

A_i denotes the amount of the i^{th} claim $(i = 1,2,\ldots)$ and $N(t)$ represents the total number of claims occuring in the time-interval $[0,t]$.

Thus, the Riemann-Stieljes integral $\int_0^t e^{kt}\, d\, S(t)$ becomes :

$$(23)\quad \sum_{i=1}^{N(t)} e^{k\, t_i} A_i$$

where $t_1, t_2, \ldots,$ denote the times at which claims occur.

The surplus process (7) is now :

$$(24)\quad Z(t) = x + \frac{p}{k}(1 - e^{-kt}) - e^{-kt} \sum_{i=1}^{N(t)} e^{kti} A_i + \sigma e^{-kt} X_t$$

or equivalently

$$(25)\quad Z(t) = e^{-kt}[x\, e^{kt} + \frac{p}{k}(e^{kt} - 1) + \sigma X_t - \sum_{i=1}^{N(t)} e^{kti} A_i]$$

$$= e^{-kt}[\widetilde{X}_t - X_t^*]$$

where

$$(26)\quad \widetilde{X}_t = x\, e^{kt} + \frac{p}{k}(e^{dt} - 1) + \sigma X_t$$

$$(27)\quad X_t^* = \sum_{i=1}^{N(t)} e^{kti} A_i$$

As before, $\widetilde{X}_t$ is a gaussian process with independent increments with $\mathbb{E}[\widetilde{X}_t] = x\, e^{kt} + \frac{p}{k}(e^{kt} - 1)$

$$\mathrm{var}[\widetilde{X}_t] = \frac{\sigma^2}{2k}(e^{2kt} - 1).$$

Consider $\widetilde{Z}(t) = Z(t) - x$

Obviously $\widetilde{Z}(t)$ is a process with independent increments, then we can apply GERBER's result [4] to calculate an upper bound for $\Psi(x,t)$.
In our case, we have

$$(28)\quad \Psi(X,t) \leqslant \min_r e^{-rx} \max_{0\leqslant s\leqslant t} \exp \mathbb{E}[e^{-r\widetilde{Z}(t)}]$$

Since $(\widetilde{X}(t),\ t\geqslant 0)$ and $(X^*(t),\ t\geqslant 0)$ are independent, (28) reduces to :

$$(29)\quad \Psi(x,t) \leqslant \min_{r} e^{-rx} \max_{0\leqslant s\leqslant t} \exp\{-r(x + \tfrac{p}{k})(e^{ks} - 1)$$

$$+ \frac{\sigma^2}{4k}(e^{2ks} - 1)r^2 + K^*(r,s)\}$$

where $K^*(r,x)$ is the cumulant generating function of X_t^*

From C.G. TAYLOR's paper [13], we have

$$(30)\quad K^*(r,s) = \frac{\lambda}{k} \int_r^{re^{kt}} \frac{\alpha(u) - 1}{u} du$$

where $\alpha(u)$ denotes the moment generating function associated with $F(.)$. As in section 4, we can only consider values of r such that $r > r_2(t)$ with $r_2(t)$ being the unique real and positive solution of

$$(31)\quad -r(x + \frac{p}{k})(e^{kt} - 1) + \frac{\sigma^2}{4k}(e^{2kt} - 1)r^2 + K^*(r,t) = 0.$$

Then

$$(32)\quad \Psi(x,t) \leqslant \min_{r\geqslant r_2(t)} \exp[-rx - r(x + \tfrac{p}{k})(e^{kt} - 1)$$

$$+ \frac{\sigma^2}{4k}(e^{2kt}-1)r^2 + K^*(r,t)]$$

Example :

if $F(x) = 1 - e^{-x}$ i.e.

negative exponential claim size distribution, then we have

$$(33)\quad K^*(r,t) = \frac{1}{k} \log \{\frac{1 - r}{1 - r\,e^{kt}}\}$$

Some numerical results will be given in the future.

Remarks :

1) When $\sigma = 0$ we have a case treated by Taylor in [13]
2) If we take for $S(t)$ a linear combination of a compound Poisson and Wiener processes (but independent), the whole analysis, in section 4 and 5 is still valid.

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