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ANALYSIS OF SINGLE SERVER QUEUEING SYSTEMS WITH VACATION PERIODS

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ABSTRACT

For an M/G/1-type queueing system with a different service time distribution for the first customer served in a busy period, we consider two types of vacation policies for the removable server and investigate the transient and steady-state behaviour of the waiting time process for the FIFO discipline. We then extend the steady-state analysis to the case where the server applies a combination of a vacation policy and the (0,k)-policy.

1. *Introduction*

In [1], Levy & Yechiali studied the steady-state of two M/G/1 models in which the removable server leaves the system for a "vacation period" whenever a service terminates with no customers left in the queue.

We show that "Model 1" and "Model 2" in [1] - extended by considering a different service time distribution for the first customer served in a busy period - are both examples of the generalized queueing system considered in [2] and [3], for which the transient behaviour of the waiting time process was studied via an algebraic approach based on the concept of Wendel projection in the case of arbitrary interarrival and service time distributions [2] and by using integral representations of the involved operators in the case of interarrival times or service times having a rational characteristic function [3]. From the results in [3], we derive the transient behaviour of the waiting time process for our extended Models 1 and 2 and, as a limit result, we get the stationary distribution of this process.

We then generalize the models by considering a combination of the vacation policy and the (0,k)-policy and extend the steady-state analysis of [1] for what the queue length concerns.

2. *Description of the models*

Customers $\mathcal{C}_n$, $n \geqslant 0$, arrive according to a Poisson process of parameter λ and are served in the order of their arrival.

We introduce the following notations relative to customer $\mathcal{C}_n$ $(n \geqslant 0)$:

T_n : the arrival instant ;

T'_n : the service initiation instant ;

T''_n : the departure instant

and define the random variables :

$$a_n = T_{n+1} - T_n \qquad \text{(interarrival time) ;}$$

$$d_n = T'_n - \max\,(T''_{n-1}, T_n) \qquad \text{(delay imposed on } \mathcal{C}_n\text{) ;}$$

$$s_n = T''_n - T'_n \qquad \text{(service time of } \mathcal{C}_n\text{) ;}$$

$$w_n = T'_n - T_n \qquad \text{(waiting time of } \mathcal{C}_n\text{) ;}$$

$$v_n = T''_n - T_n \qquad \text{(sojourn time of } \mathcal{C}_n\text{),}$$

for which the following relations hold :

$$\left.\begin{aligned} v_n &= [\,v_{n-1} - a_{n-1}]^+ + d_n + s_n \\ w_n &= [\,v_{n-1} - a_{n-1}]^+ + d_n \end{aligned}\right\} \qquad (n \geqslant 1)$$

For both models, we have (for $n \geqslant 1$) :

$$s_n = \begin{cases} {}_o s_n & \text{if } T_n \leqslant T''_{n-1} \text{ ;} \\ {}_1 s_n & \text{if } T_n > T''_{n-1} \end{cases}$$

where $\{({}_o s_n, {}_1 s_n)\}_{n\geqslant 1}$ is a sequence of i.i.d. random vectors, independent of $\{a_n\}_{n\geqslant o}$.

For Model 1 :

$$d_n = \begin{cases} 0 & \text{if } T_n \leqslant T''_{n-1} \text{ or } T_n > T''_{n-1} + u_n \text{ ;} \\ T''_{n-1} + u_n - T_n & \text{if } T''_{n-1} < T_n \leqslant T''_{n-1} + u_n, \end{cases}$$

where $\{u_n\}_{n\geqslant 1}$ is a sequence of i.i.d. random variables, independent of $\{({}_o s_n, {}_1 s_n, a_{n-1})\}_{n\geqslant 1}$.

(u_n is the duration of the single vacation during which the server leaves the system if no customers are left in the queue at T''_{n-1})

For Model 2 :

$$d_n = \begin{cases} 0 & \text{if } T_n \leqslant T''_{n-1} ; \\ T''_{n-1} + \sum_{\nu=1}^{i} u_{n,\nu} - T_n & \text{if } T''_{n-1} + \sum_{\nu=1}^{i-1} u_{n,\nu} < T_n \leqslant T''_{n-1} + \sum_{\nu=1}^{i} u_{n,\nu} \\ & (i \geqslant 1) \end{cases}$$

where the $u_{n,\nu}$ $(n \geqslant 1,\ \nu \geqslant 1)$ are i.i.d. random variables independent of $\{({}_0s_n,\ {}_1s_n,\ a_{n-1})\}_{n\geqslant1}$.

($u_{n,1}$, $u_{n,2}$, ... are the durations of the successive vacations during which the server leaves the system if no customers are left in the queue at T''_{n-1}, until, coming back from such a vacation, he finds a non empty queue)

3. *Transient and steady-state behaviour of the sojourn time and the waiting time processes*

Using the fact that the arrival process is a Poisson process, it can be derived from the above description that :

$$d_n = \begin{cases} 0 & \text{if} \quad T_n \leqslant T''_{n-1} ; \\ {}_1d_n & \text{if} \quad T_n > T''_{n-1}, \end{cases}$$

where $\{{}_1d_n\}_{n\geqslant1}$ is a sequence of i.i.d. random variables independent of $\{({}_0s_n,\ {}_1s_n,\ a_{n-1})\}_{n\geqslant1}$, the common distribution of which is given by :

for Model 1 :

$$E\,(e^{-\theta\ {}_1d_n}) = \frac{1}{\theta - \lambda}\,[\theta\ u\ (\lambda) - \lambda\ u\ (\theta)] \qquad (1)$$

for Model 2 :

$$E\,(e^{-\theta\ {}_1d_n}) = \frac{1}{1 - u\ (\lambda)}\ \frac{\lambda}{\theta - \lambda}\,[u\ (\lambda) - u\ (\theta)], \qquad (2)$$

where $u\ (\theta)$ denotes either $E\,(e^{-\theta\ u_n})$ or $E\,(e^{-\theta\ u_{n,\nu}})$.

Thus either model is an example of the generalized queueing system considered in [2] and [3].

Supposing that v_o is independent of the vectors $({}_1d_n, {}_os_n, {}_1s_n, a_{n-1})$, $n \geqslant 1$, and using the following notations :

$$h_o(\theta) = E(e^{-\theta v_o}) \; ; \; g_o(\theta) = E(e^{-\theta w_o})$$

$$h_{(z)}(\theta) = \sum_{n \geqslant 0} z^n E(e^{-\theta v_n}) \; ; \; g_{(z)}(\theta) = \sum_{n \geqslant 0} z^n E(e^{-\theta w_n})$$

$$(|z| < 1)$$

$${}_1d(\theta) = E(e^{-\theta\, {}_1d_n}) \; ; \; {}_is(\theta) = E(e^{-\theta\, {}_is_n}) \qquad (i = 1, 2),$$

we deduce from the results in [3] that :

$$h_{(z)}(\theta) = \frac{1}{\theta - \lambda + z\lambda\, {}_os(\theta)} \{(\theta - \lambda)\, h_o(\theta) + z\,\xi_{(z)}\, [\lambda\, {}_os(\theta) + (\theta - \lambda)\, {}_1d(\theta)\, {}_1s(\theta)]\}$$

$$g_{(z)}(\theta) = \frac{1}{\theta - \lambda + z\lambda\, {}_os(\theta)} \{(\theta - \lambda)\, g_o(\theta) + \lambda z\, [{}_os(\theta)\, g_o(\theta) - h_o(\theta)] + z\xi_{(z)}[\lambda + (\theta - \lambda)\, {}_1d(\theta) + z\lambda\, {}_1d(\theta)\, ({}_os(\theta) - {}_1s(\theta))]\}$$

with $$\xi_{(z)} = \frac{h_o(\varepsilon(z))}{1 - z\, {}_1d(\varepsilon(z))\, {}_1s(\varepsilon(z))}$$

where, for $|z| < 1$, $\varepsilon(z)$ denotes the unique zero of the function $\theta - \lambda + z\lambda\, {}_os(\theta)$ in the half-plan $\mathrm{Re}\,\theta > 0$.
The function ${}_1d(\theta)$ is given by (1) and (2) for Models 1 and 2 respectively.

Putting $\bar{u} = E(u_n) = E(u_{n,\nu})$

$${}_1\bar{d} = E({}_1d_n) \; ; \; {}_i\bar{s} = E({}_is_n) \quad (i = 1, 2)$$

and supposing that these expectations are finite and that $\lambda\ {}_o\bar{s} < 1$, one gets, for the stationary distributions

$$h(\theta) = \lim_{z\to 1} (1-z)\, h_{(z)}(\theta)\ ;\ g(\theta) = \lim_{z\to 1} (1-z)\, g_{(z)}(\theta) :$$

$$h(\theta) = \xi\, \frac{\lambda\ {}_o s(\theta) + (\theta-\lambda)\ {}_1 d(\theta)\ {}_1 s(\theta)}{\theta - \lambda + \lambda\ {}_o s(\theta)} \qquad (3)$$

$$g(\theta) = \xi\, \frac{\lambda + (\theta-\lambda)\ {}_1 d(\theta) + \lambda\ {}_1 d(\theta)\,({}_o s(\theta) - {}_1 s(\theta))}{\theta - \lambda + \lambda\ {}_o s(\theta)}$$

with $$\xi = \frac{1 - \lambda\ {}_o\bar{s}}{1 - \lambda\,({}_o\bar{s} - {}_1\bar{s}) + \lambda\ {}_1\bar{d}}$$

For Model 1, ${}_1 d(\theta)$ is given by (1) and ${}_1\bar{d} = \dfrac{\lambda\,\bar{u} + u(\lambda) - 1}{\lambda}$

For Model 2, ${}_1 d(\theta)$ is given by (2) and ${}_1\bar{d} = \dfrac{\lambda\,\bar{u} + u(\lambda) - 1}{\lambda\,(1 - u(\lambda))}$

Particularizing relation (3) to the case where ${}_o s(\theta) \equiv {}_1 s(\theta)$, one gets the expressions given in [1] for the steady-state distribution of the sojourn time in Models 1 and 2.

4. *Combination of the vacation policies with the (0,k)-policy*

In this section, we generalize the models described in section 2 by combining the vacation policy with the (o,k)-policy ($k \geqslant 1$) i.e. :

<u>for Model 1</u> : when a service terminates with no customers left in the system, the server leaves for a single vacation. When coming back, he immediately initiates a busy period if at least k customers are queueing ; otherwise, he waits until k customers are present to start serving again ;

<u>for Model 2</u> : when a service terminates with no customers left in the system, the server leaves for successive vacations, until, coming back from such a vacation, he finds at least k customers queueing.

The steady-state analysis performed in [1] for $k = 1$ can be readily extended. We consider the instants $\tau_1, \tau_2, \ldots, \tau_n, \ldots$ at which either a service or a vacation period terminates (where by vacation period, we mean a single vacation for Model 1 and a sequence of successive vacations for Model 2) and we define $x_n = (i_n, l_n)$, where l_n denotes the queue length at $\tau_n + 0$ and i_n is 0 (respectively 1) if a vacation period (respectively a service) terminates at τ_n.

Putting

$$p^{(k)}_{j|1} = \lim_{n\to\infty} \Pr\,[\,l_n = j \,|\, i_n = 1]$$

and supposing that $\lambda\,{}_o\bar{s} < 1$, we get the following expression for the generating function $P^{(k)}_{|1}(z) = \sum_{j\geq 0} z^j\, p^{(k)}_{j|1}$ $(|z| \leq 1)$:

- for Model 1

$$P^{(k)}_{|1}(z) = \xi^{(k)}_1 \frac{{}_1s\,(\lambda(1-z))\,[\,u(\lambda(1-z)) + \sum_{r=0}^{k-1} b_r\,(z^k - z^r)] - {}_os\,(\lambda(1-z))}{z - {}_os\,(\lambda(1-z))}$$

with $$\xi^{(k)}_1 = \frac{1 - \lambda\,{}_o\bar{s}}{\lambda\,\bar{u} + \sum_{r=0}^{k-1} b_r\,(k-r) - \lambda\,({}_o\bar{s} - {}_1\bar{s})}$$

where $$b_r = \int_o^\infty e^{-\lambda t}\,\frac{(\lambda\, t)^r}{r!}\, d\,\Pr\,[\,u_n \leq t] \qquad (r \geq o)$$

- for Model 2

$$P^{(k)}_{|1}(z) = \xi^{(k)}_2 \frac{{}_1s\,(\lambda(1-z))\,\tilde{B}^{(k)}(z) - {}_os\,(\lambda(1-z))}{z - {}_os\,(\lambda(1-z))} \qquad (4)$$

with $$\xi^{(k)}_2 = \frac{1 - \lambda\,{}_o\bar{s}}{\overline{\tilde{n}^{(k)}} - \lambda\,({}_o\bar{s} - {}_1\bar{s})}$$

where $$\tilde{B}^{(k)}(z) = \sum_{r\geq k} z^r \Pr\,[\,\tilde{n}^{(k)} = r]$$

$$\overline{\tilde{n}^{(k)}} = E\,(\tilde{n}^{(k)}),$$

$\tilde{n}^{(k)}$ denoting the number of customers arriving during a vacation period.

The $\tilde{B}^{(k)}(z)$ and $\overline{\tilde{n}^{(k)}}$, $k \geqslant 1$, can be recursively derived using the following relations :

$$\tilde{B}^{(k)}(z)\ (1 - b_o) = \sum_{r=1}^{k-1} b_r\ z^r\ [\tilde{B}^{(k-r)}(z) - 1] + u\ (\lambda(1-z)) - b_o \qquad (5)$$

$$\overline{\tilde{n}^{(k)}}\ (1 - b_o) = \sum_{r=1}^{k-1} b_r\ \overline{\tilde{n}^{(k-r)}} + \lambda\ \bar{u} \qquad (6)$$

Remark : a further extension

Model 2 can be further extended by allowing the distribution of the duration $u_{n,\nu}$ of the ν^e vacation in a vacation period to depend on the number of customers present in the system when the vacation begins : given that this number is equal to r ($r = 0, \ldots, k-1$), the conditional distribution of $u_{n,\nu}$ has the Laplace-Stieltjes transform $u_r\ (\theta)$ (we put $u_0\ (\theta) \equiv u\ (\theta)$). [This generalization was suggested to me by Jacques Teghem Jr.]

Relation (4) still applies, as well as relations (5) and (6), the random variable $\tilde{n}^{(k-r)}$ corresponding now to the model combining the (0, k-r)-policy and the vacation policy in which the conditional distribution of a vacation, given that r' customers are present when it begins, has Laplace-Stieltjes transform $u_{r'+r}\ (\theta)$ ($r' = 0, \ldots, k-r-1$).

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