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ON AN OPTIMAL POLICY FOR DIVERTING TRAFFIC FLOW FROM A CONGESTED AREA

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ABSTRACT

This work investigates policies for diverting traffic flow from a main way where some flow-stopping incident has occurred. The model chosen for describing the main way congestion is basically a queuing model: vehicles that are trapped by the accident can leave the jam, but at a slower than normal rate, and the congestion will terminate when the waiting queue becomes empty.

The new feature introduced is that there exists a branching point in the upstream of the congested area that gives a controller (either human or automatic) the ability of diverting a fraction of vehicular flow towards some uncongestioned auxilary way. The objective aimed is to minimize a cost function that measures 1) the amplitude of the congestion as the total number of vehicles involved in the jam, the jam duration, and the total vehicle-hours waited, and 2) diversion costs that may take into account the lengthening in travel time incurred by diverted drivers.

Traffic diversion policies are analyzed by using a Markov (birth-and-death) model. It is shown that the best rule leads simply to divert an arriving vehicle if and only if the current queue length exceeds some given upper limit.

INTRODUCTION

Decisional models for vehicular traffic generally deal with routing a traffic flow (viewed as either discrete or continous) in a network [01, 02, 03,...] , or with synchronizing a series of traffic lights [04, 05, 06,...] . These models usually consider normal (i.e. expected) conditions on traffic flow, and analyze steady-state behavior of traffic systems.

We examine in this work another kind of situations which, while "accidental", may be nevertheless of some interest : that concerns rules for diverting a vehicular flow from a main way where some flow-stopping incident has occurred. Among queuing models that describe the resulting jam, we retain that of Gaver [07] : as far as traffic is concerned, - and only as far as traffic is concerned -, a car accident mainly results in a physical obstacle; the latter limits traffic flow to a slower than normal rate, because jammed cars get in each others way. The congestion is considered as completely dissipated when the jam queue length falls below some non-congestion level. Traffic can then flow freely again.

We introduce a decisional aspect to the problem by assuming that this stoppage happens in the context described by map (figure) 1.

Whenever a car arrives at the branching point, one has to decide whether this car should be diverted or not. Such a dichotomic choice, that should use control informations about the current "state" of the system, must take into account the following two bjectives : 1) reduction of ineffectiveness of the congestion, .nd 2) reduction of diversion costs.

As measures of the first kind, we consider :

- the jam duration, D,
- the total number of trapped (i.e. non-diverted) cars, N,
- the total vehicle-hours waited, that is the total waiting delay of trapped cars, W.

Concerning diversion costs, we assume that any diverted driver incurs the same lengthening in his travel time, hence a penalty proportional to the total number M of diverted cars.

Optimal diversion policies are those minimizing expectation of

$$C = \gamma_d D + \gamma_w W + \gamma_n N + \gamma_m M, \qquad (1)$$

where the γ's are given non-negative numbers.

A STOCHASTIC MODEL FOR CONTROLLING THE JAM DISSIPATION

Let n_1 denote the number of cars present at the jam at the beginning of the congestion, and n_2 the non-saturation level below which traffic can flow freely and the jam is completely dissipated.

We assume that during this phase vehicles arrive at the intersection according to a time-stationary Poisson process of rate λ, and that trapped vehicles leave the jam according to a "death" process of rates $\{\mu_i\}$; that is, if the jam queue length at some time t is i ($i \geqslant n_2$), then, independently of the "past history" :

$$\Pr[\,0 \text{ departure during } [t, t+\Delta t]\,] = 1 - \mu_i \Delta t + o(\Delta t),$$

$$\Pr[\,1 \text{ departure during } [t, t+\Delta t]\,] = \mu_i \Delta t + o(\Delta t),$$

$$\Pr[\,> 1 \text{ depart. during } [t, t+\Delta t]\,] = o(\Delta t),$$

for any $\Delta t \geqslant 0$. Allowing the departure process to be state-dependent, we can describe situations in which trapped cars get in each others way, by assuming for example that the sequence $\{\mu_i / i\}$ decreases to 0 as $i \to \infty$.

Since arrival and departure processes are markovian and time-stationary, we restrict ourselves to diversion policies specified by :

$$\delta = \{\delta_i ;\ i \geqslant n_2\}, \tag{2}$$

such that :

$$\delta_i \in [0,1],\ \forall\ i. \tag{3}$$

Use of policy δ means that, whenever a vehicle arrives at the intersection while i other ones are present in the jam queue, then that vehicle is diverted with probability δ_i (and non-diverted with probability $1 - \delta_i$).

For $n_1 \geq n_2$, the congestion may be considered as a period for the first passage of the jam queue length from n_1 to $n_2 - 1$. Hence, the jam is decomposable into a juxtaposition of $n_1 - n_2 + 1$ periods, period i ($i = n_2, \ldots, n_1$) being that for the first passage of the queue from i to i - 1 cars, see Figure 2.

Assume a given diversion policy is being used. We denote (i) the duration of a period i, (ii) the total time loss during this period, (iii) the number of cars that join the jam during this period, and (iv) the number of cars diverted during this period, by D_i, W_i, N_i, and M_i respectively. The total cost incurred during the decongestion phase is the sum of costs related to each of the above-mentioned periods :

$$C = \sum_{i=n_2}^{n_1} C_i, \text{ where} \tag{4}$$

$$C_i = \gamma_d D_i + \gamma_w W_i + \gamma_n N_i + \gamma_m M_i \tag{5}$$

are statistically independent of each other, by our markovian assumptions. We first derive a recursive formula for the joint Laplace transforms

$$\Phi_i(d,w,x,y) = \mathcal{E}[e^{-dD_i}.e^{-wW_i}.x^{N_i}.y^{M_i}], \tag{6}$$

$$i \geq n_2,\ d,w \geq 0,\ |x|,|y| \leq 1,$$

before investigating properties of optimal policies.

Consider now the evolution of the jam queue at some time epoch. The next "event" that will occur is either a departure from the jam or an arrival to the intersection (in the later case, the arriving car may be diverted or not). Hence, for $i \geq n_2$:

(D_i, W_i, N_i, M_i)

$$\doteq \begin{cases} (T_i, iT_i, 0,0), & \text{with probability } \mu_i/(\lambda+\mu_i), \\ (T_i', iT_i', 0,1) + (D_i', W_i', N_i', M_i'), & \text{with probability } \delta_i\lambda/(\lambda+\mu_i), \\ (T_i'', iT_i'', 1,0) + (D_{i+1}'', W_{i+1}'', N_{i+1}'', M_{i+1}'') + (D_i'', W_i'', N_i'', M_i''), & \\ \qquad \text{with probability } (1-\delta_i)\lambda/(\lambda+\mu_i), & \end{cases}$$

where the symbol $\doteq$ means equality of distributions, and where, by our markovian assumptions :

- $T_i \doteq T_i' \doteq T_i'' \doteq$ a random variable distributed exponentially with mean $(\lambda+\mu_i)^{-1}$,
- $(D_i', W_i', N_i', M_i') \doteq (D_i, W_i, N_i, M_i)$, and is statistically independent of T_i',
- $(D_{i+1}'', W_{i+1}'', N_{i+1}'', M_{i+1}'') \doteq (D_{i+1}, W_{i+1}, N_{i+1}, M_{i+1})$, and is independent of T_i'' and of

$(D_i'', W_i'', N_i'', M_i'') \doteq (D_i, W_i, N_i, M_i)$.

Hence, using the fact that the Laplace transform of (the distribution function of) T_i is :

$$\mathcal{E}[e^{-tT_i}] = (\lambda+\mu_i)/(t+\lambda+\mu_i), \quad t \geqslant 0,$$

the above inductive decomposition of (D_i, W_i, N_i, M_i) leads readily to:

$$\Phi_i(d,w,x,y) = \frac{\mu_i + \delta_i\lambda y\Phi_i(d,w,x,y) + (1-\delta_i)\lambda x\,\Phi_{i+1}(d,w,x,y)\Phi_i(d,w,x,y)}{d + iw + \lambda + \mu_i} \tag{7}$$

This formula links Φ_i and Φ_{i+1}, and allows recursive computation of Φ_i, $\forall$ i, provided some Φ_j is known :

$$\ldots \leftarrow \Phi_{j-2} \leftarrow \Phi_{j-1} \leftarrow \Phi_j \rightarrow \Phi_{j+1} \rightarrow \Phi_{j+2} \rightarrow \ldots$$

It will be especially so if some $\delta_j = 1$, that is, if we decide to divert any vehicle that arrives while the jam size is currently j. In this case, as one can easily anticipate, Φ_j no longer depends on Φ_{j+1}, and the above formula reduces to a linear equation in Φ_j whose solution is :

$$\Phi^1_j(d,w,x,y) \;=\; \mu_j \;/\; (d + jw + \lambda + \mu_j - \lambda y). \tag{8}$$

Note that, from a practical point of view, there generally exists a constraint of capacity on the main way that implies automatic diversion whenever the jam size reaches some oversaturation level. We now show that, even when such a constraint is not imposed, the best diversion rule merely consists in diverting an arriving car if and only if the current queue length reaches some *diversion level*.

Recall that optimal diversion policies are those minimizing(+) :

$$\bar{C} \;=\; \sum_{i=n_1}^{n_2} \bar{C}_i. \tag{9}$$

Taking partial derivatives of Φ_i, we obtain from (7) :

$$\begin{aligned}
\bar{D}_i &= \{1 + (1-\delta_i)\lambda\bar{D}_{i+1}\} \,/\, \mu_i\,,\\
\bar{W}_i &= \{i + (1-\delta_i)\lambda\bar{W}_{i+1}\} \,/\, \mu_i\,,\\
\bar{N}_i &= \{(1-\delta_i)\lambda + (1-\delta_i)\lambda\bar{N}_{i+1}\} \,/\, \mu_i\,,\\
\bar{M}_i &= \{\delta_i\lambda + (1-\delta_i)\lambda\bar{M}_{i+1}\} \,/\, \mu_i\,.
\end{aligned} \tag{10}$$

(+) Subsequently, upper bar denotes mathematical expectation.

Henceforth, from (5) and (10) :

$$\bar{C}_i = \frac{(1-\delta_i)\lambda}{\mu_i} (\bar{C}_{i+1} + \gamma_n - \gamma_m) + \bar{C}'_i, \tag{11}$$

where $\bar{C}'_i$ is the cost incurred on the average during period i when $\delta_i = 1$:

$$\bar{C}'_i = (\gamma_d + i\gamma_w + \lambda\gamma_m) / \mu_i. \tag{12}$$

Observe that the $\bar{C}'_i$'s are "constant", that is, independent of δ; they are also readily computable from parameters of the problem. Equation (11) holds for any diversion policy, hence also for an optimal one; denoting such a policy by $\tilde{\delta}$, and the corresponding costs by $\tilde{C}_i$, we then have :

$$\tilde{C}_i = \frac{(1-\tilde{\delta}_i)\lambda}{\mu_i} (\tilde{C}_{i+1} + \gamma_n - \gamma_m) + \bar{C}'_i. \tag{$\widetilde{11}$}$$

From this last equation, it is obvious that :

- if $(\tilde{C}_{i+1} + \gamma_n - \gamma_m) > 0$, then we must have $\tilde{\delta}_i = 1$, while
- if $(\tilde{C}_{i+1} + \gamma_n - \gamma_m) < 0$, then $\tilde{\delta}_i = 0$.

(This can also be justified by the principle of optimality: a car that arrives while i other ones are present causes a marginal cost of γ_m if it is diverted, and a cost $\tilde{C}_{i+1} + \gamma_n$ if it is not diverted). Ignoring the cases where $(\tilde{C}_{i+1} + \gamma_n - \gamma_m) = 0$, (in these cases any value of δ_i is optimal),we may say that optimal diversion policies satisfy a property of all-or-nothing one encounters in other diversion assignment problems [01].

From ($\widetilde{11}$), it is obvious that, since the $\tilde{C}_i$ are related to an optimal policy while the $\bar{C}'_i$ concern a non-necessarily optimal one :

$$\tilde{C}_i \leqslant \bar{C}'_i \quad , \quad \forall\ i.$$

Hence : $(\bar{C}'_{i+1} + \gamma_n - \gamma_m) < 0$ implies, *a fortiori* : $(\tilde{C}_{i+1} + \gamma_n - \gamma_m) < 0$.

This leads to the following practical rule :

Proposition 1.

| If $\bar{C}'_{i+1} < \gamma_m - \gamma_n$, then $\tilde{\delta}_i = 0$.

Another case where a trivial solution to our optimization problem exists is that $\gamma_m - \gamma_n \leqslant 0$. Since $\tilde{C}_{i+1} > 0$, we then have :

Proposition 2.

| If $\gamma_m - \gamma_n \leqslant 0$, then $\tilde{\delta}_i = 1$, $\forall$ i.

In the remaining part of this section, we shall examine the case :

$$\gamma_m - \gamma_n > 0. \tag{13}$$

Proposition 3.

| $\tilde{C}_i \geqslant (\gamma_d + i\gamma_w + \lambda\gamma_n) / \mu_i$, $\forall$ i.

Proof.

$$\begin{aligned}
\tilde{C}_i &= \min_{\delta} \bar{C}_i \quad \text{(this minimum is taken under the constraints (2,11,12)}\\
&\geqslant \min_{\delta_i} \frac{(1-\delta_i)\lambda}{\mu_i} \tilde{C}_{i+1} + \min_{\delta_i} \frac{(1-\delta_i)\lambda}{\mu_i} (\gamma_n - \gamma_m) + \bar{C}'_i \text{ , by (11)}\\
&= 0 - \frac{\lambda}{\mu_i} (\gamma_m - \gamma_n) + \bar{C}'_i \text{ , by (13)}\\
&= (\gamma_d + i\gamma_w + \lambda\gamma_n) / \mu_i\text{, by (12)} \blacksquare
\end{aligned}$$

Propositions 1 and 3 are now used to establish

Theorem 1.

| Assume that the sequence $\{(\gamma_d + i\gamma_w + \lambda\gamma_n) / \mu_i ; i \geqslant n_2\}$ increases monotonously to infinity with i (hence the sequence $\{\bar{C}'_i\}$ also does).

Let :

$$i_d = \max \{i \ni \bar{C}'_{i+1} < \gamma_m - \gamma_n\}. \tag{14}$$

Then there exists an optimal diversion policy $\tilde{\delta}$ that satisfies :

$$\tilde{\delta}_i = \begin{cases} 0, \text{ for } i \leqslant i_d, & (15)\\ 1, \text{ for } i > i_d. & (16)\end{cases}$$

The basic assumption of this theorem can be justified as follows : at any time, the more there are trapped cars, the more they obstruct each other and the lower is the instantanous flow rate ($\mu_i \to 0$); in any way, this auto-obstruction implies that jam dissipation rates cannot be higher than that of non-congested M/M/∞ queues ($\mu_i = i\mu$); therefore μ_i never increases more rapidly than i ($\mu_i / i \to 0$).

Proof.

Equation (15) is a mere restatement of Proposition 1. For proving (16), we proceed by contradiction and assume that :

$$\tilde{\delta}_j \neq 1, \text{ for some } j > i_d , \tag{17}$$

for any optimal policy $\tilde{\delta}$. Then we show that :

$$\tilde{\delta}_k \neq 1, \forall\, k \geqslant j. \tag{18}$$

But, using ($\tilde{11}$) with i = k, inequations (18) imply :

$$\tilde{C}_{k+1} + \gamma_n - \gamma_m \leqslant 0, \forall\, k \geqslant j \tag{19}$$

(otherwise, $\tilde{C}_{k+1} + \gamma_n - \gamma_m > 0$, and $\tilde{\delta}_k$ would be 1). This contradicts our basic assumption and Proposition 3, since $\tilde{C}_i \to \infty$ as $i \to \infty$.

It remains to establish (18), by induction.

If for some $k \geqslant i_d$: $\tilde{\delta}_k \neq 1$ for any optimal policy $\tilde{\delta}$, then :

$$\tilde{C}_{k+1} + \gamma_n - \gamma_m \leqslant 0.$$

But the case $\tilde{C}_{k+1} + \gamma_n - \gamma_m = 0$ is to be discarded, since $\tilde{\delta}_k$ may be chosen equal to 1. Therefore :

$$\tilde{C}_{k+1} + \gamma_n - \gamma_m < 0. \tag{20}$$

On the other hand, by definition (14) of i_d :

$$\bar{C}'_{k+1} \geqslant \gamma_m - \gamma_n. \tag{21}$$

Inequalities (20) and (21) imply :

$$\bar{C}'_{k+1} > \tilde{C}_{k+1}.$$

Reusing ($\tilde{11}$) with i = k+1 : $\tilde{\delta}_{k+1} \neq 1$ (otherwise, $\tilde{\delta}_{k+1} = 1$, and we would have $\tilde{C}_{k+1} = \bar{C}'_{k+1}$) ■

CONCLUDING REMARKS

Theorem 1 provides a simple and practical rule for vehicular diversion since one can readily obtain the optimal *diversion level* from parameters of the congestion, and then use the policy defined by (15, 16). May be these parameters, and especially the jam dissipation rates, are not easy to estimate in practice, and this difficulty may limit the useness of the theorem. However, the kind of policy it states, – divert above some jam size –, seems very reasonable under natural conditions of auto-obstruction. Intuitively, such a result should hold also under more general hypotheses about arrival and departure processes. Note that the imbedded decision process owns the following trivial characteristic : choices must be made at and only at car arrival epochs. Therefore, even if the arrival process is "general" (that is, a renewal process) but the departure process is kept markovian, then the diversion decisional process remains markovian too, meaning that jam size at arrival times still constitutes the exhaustive information to our control problem.

Under the above stochastic assumptions, a recursive formula for $\bar{C}_i$, in an integral form, is not difficult to obtain. Unfortunately, we are not able to derive from it a generalization of Theorem 1.

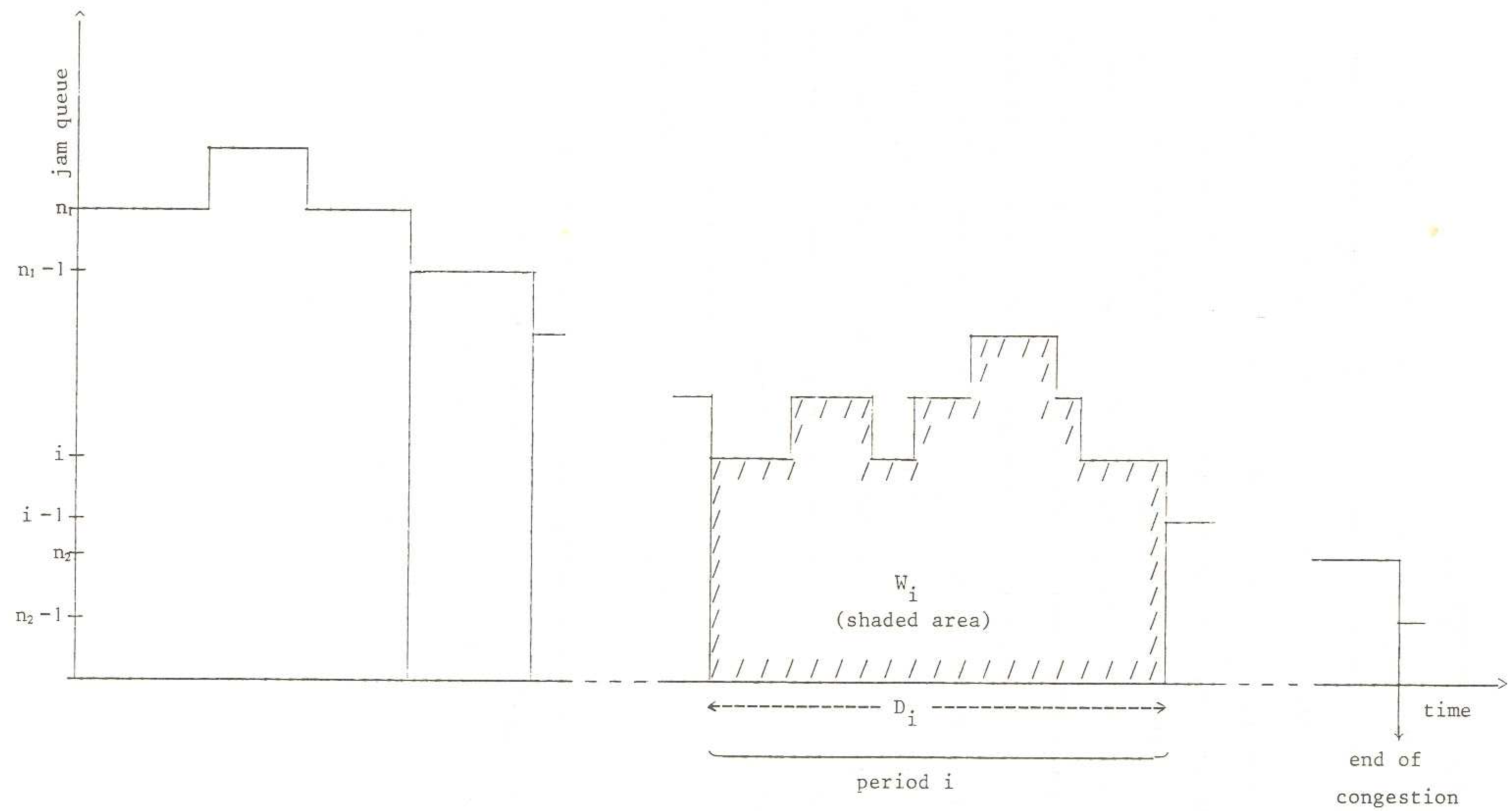

Figure 2. Decomposition of jam dissipation.

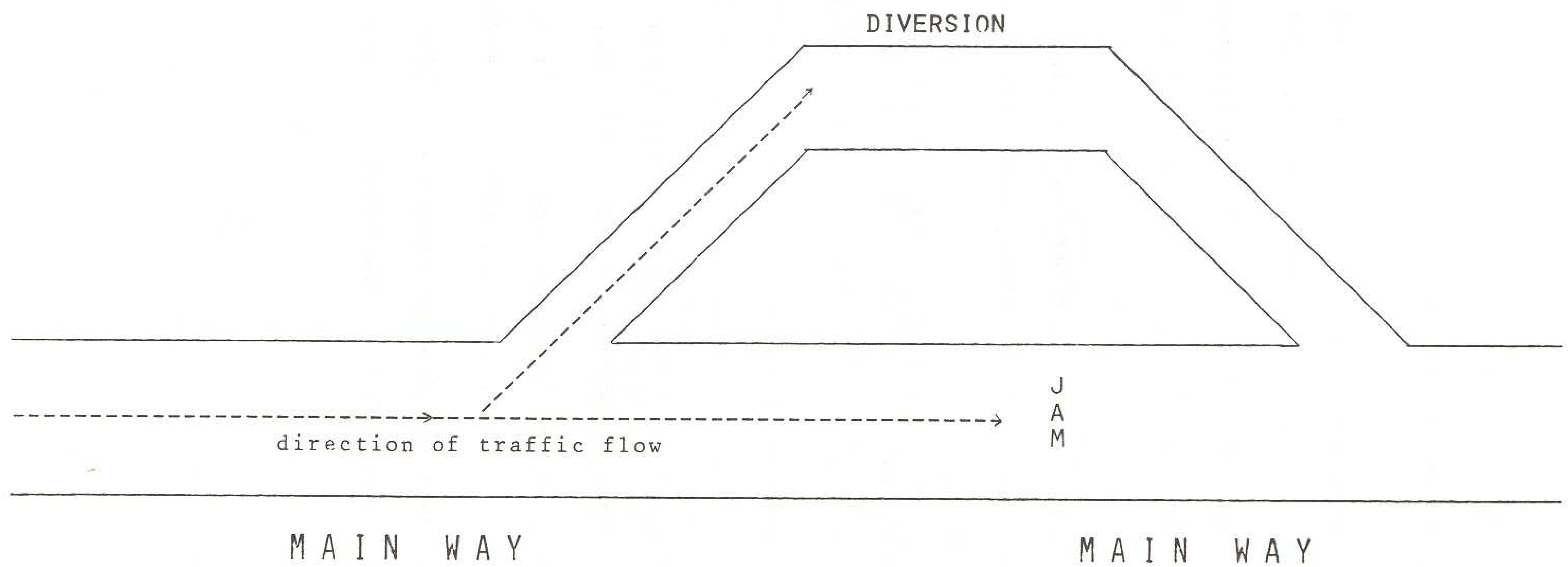

Figure 1. Map of the congested area.

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