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A NEW ALGORITHM FOR RESISTANT REGRESSION

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ABSTRACT

The ordinary least squares regression method is not a reliable tool in regression analysis without first diagnosing possible outliers present in the data set. The least median of squares regression technique (Rousseeuw 1984), which is designed to lessen the impact of outlying observations, is presented and some alternatives are given. The output of a Fortran implementation of this regression technique, called PROGRESS (Leroy and Rousseeuw 1984), is illustrated with an example. The results can be interpreted by means of graphical representation of the standardized residuals. It is showed how PROGRESS can be used as a diagnostic tool in regression analysis.

1. INTRODUCTION

In a linear model, a response variable y is written as a linear combination of p explanatory variables $x_1,\ldots,x_p$

$$y_i = x_{1i}\theta_1 + \ldots + x_{pi}\theta_p + e_i \ , \qquad i=1,\ldots,n$$

where e_i is often assumed to be normally distributed with mean zero and standard deviation σ. Until recently, most people have been estimating the coefficients $\theta_1,\ldots,\theta_p$ almost exclusively by means of the least squares (LS) method, defined by

$$\underset{\hat{\theta}}{\text{minimize}} \quad \sum_{i=1}^{n} r_i^2 \qquad (1.1)$$

where the residuals r_i equal $y_i - x_{1i}\hat{\theta}_1 - \ldots x_{pi}\hat{\theta}_p$.

The main advantage of this method lies in the fact that explicit formulas exist for the estimates, making it the only feasible method in the pre-computer age. For the same reason, nowadays many computer programs for LS are available, which explains why this method has been used so often. Moreover, many mathematicians adore the LS estimator because of its nice optimality properties under the condition of a normal error structure. In practical situations however, this condition is hardly fulfilled, and the LS regression technique is quite sensitive to the presence of outlying points. Therefore, it is important to have a diagnostic tool for identifying such points. In the last decades, several statisticians have given consideration to robust regression (see Rousseeuw 1984 for an overview) on the one hand and to regression diagnostics on the other hand. Both approaches are closely related by two important common aims, namely identifying outliers and pointing out inadequacies of the model. The books of Belsley, Kuh and Welsch (1980) and Cook and Weisberg (1982) are dedicated to regression diagnostics. However, most of these methods deal with the effects of deleting a single point, and often do not succeed in identifying multiple outliers. On the other hand, the robust regression technique described in this paper does manage to

solve this problem. When the robust and the LS fit differ substantially, this indicates that the data require a thoughtful analysis.

In order to express in a statistical way the robustness of a regression technique against outlying observations, Hampel(1971,1975) proposed a general asymptotic definition of the breakdown point ϵ^*. We will use the finite sample version of this notion given by Donoho and Huber (1982), namely

$$\epsilon^*(X,T) = \min \{m/n \; ; \quad \sup \|T(X')\| = \infty\}$$

where the X' are obtained by replacing m points of the sample X (containing n data points) by arbitrary ones. T stands for a regression estimator. In words, ϵ^* is the smallest fraction of contamination that can cause the estimates to take on arbitrarily large values. For LS regression ϵ^* equals 1/n because one bad point is sufficient to carry the LS estimator over all bounds. Considering the limit for n going to infinity (p fixed), one can establish that LS has ϵ^* equal to 0%. The best possible value for the breakdown point is 50%, because for larger amounts of contaminated data in a sample, one cannot tell the 'good' and the 'bad' observations apart. The first regression estimator which is equivariant for linear transformations on the x_i and which attains a breakdown point of 50% is the least median of squares (LMS) estimator (Rousseeuw 1984).

The LMS estimate of θ corresponds to

$$\underset{\hat{\theta}}{\text{minimize}} \; \underset{i}{\text{median}} \; r_i^2 \qquad (1.2)$$

Compared to LS (1.1), the sum has been replaced by the median. Preceding improvements towards robustness consisted of substituting the square by something else, but none of these led to a high breakdown point.

In the following two sections we will outline the algorithm we use for computing the LMS estimator as well as some other robust regression estimates derived from it. Section 4 is devoted to an example.

2. ALGORITHM FOR COMPUTING THE LMS ESTIMATES

The special case of one-dimensional estimation of location is obtained by putting $p=1$ and $x_i=1$ for all i in (1.2). Then the minimization becomes

$$\underset{\hat{\theta}}{\text{minimize}} \ \text{median}_i \ (y_i - \hat{\theta})^2 \qquad (2.1)$$

and the sample reduces to $(y_i)_{i=1,\ldots,n}$. The LMS estimate is then equal to the midpoint of the shortest half of the sample $(y_i)_{i=1,\ldots,n}$. The shortest half is given by the smallest of the differences $y_{(h)}-y_{(1)}$, $y_{(h+1)}-y_{(2)}$, $\ldots$, $y_{(n)}-y_{(n-h+1)}$, where $h=[n/2]+1$ ($[x]$ means integer part of x), and $y_{(1)} \leq \ldots \leq y_{(n)}$ are the ordered observations.

The following simple example will illustrate the LMS estimate. Consider the one-dimensional sample consisting of the observations:

21, 23, 25, 26, 26, 299.

The halves of this sample are indicated by the lines below the values. The LMS estimate of location is 24.5, because it is the midpoint of the shortest half. The least squares estimate of location is the mean, which equals 70 in this sample. Comparing both estimates, it appears that 24.5 is a better parameter of location for the majority of the data. The aberrant value 299 has badly affected the mean, whereas the LMS has completely neglected its presence.

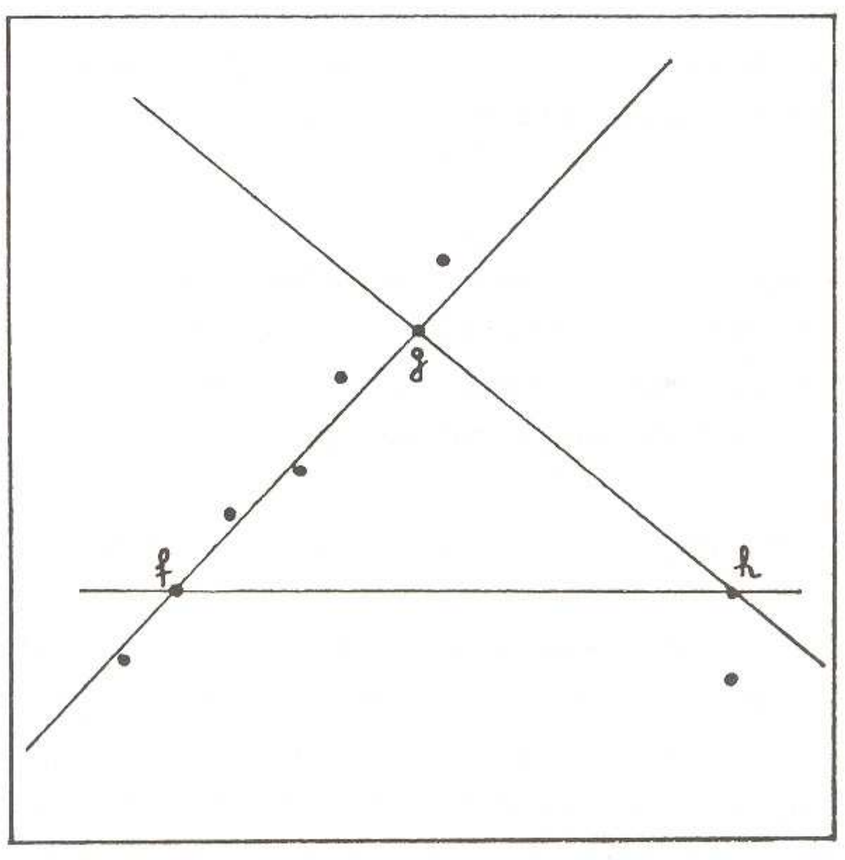
In the general regression model, it is probably not possible to write down a straightforward formula for the LMS estimate. For this case we have therefore constructed a heuristic algorithm which can be outlined in the following way:
Choose at random p observations out of the n and determine the unique regression surface through these p points. This solution gives a trial estimate $(\theta_1^\circ, \ldots, \theta_p^\circ)$. This procedure is repeated m times and the trial estimate for

which the objective function is minimal is retained. The number of replications (m) is determined by requiring that the probability that at least one of the m subsamples is 'good' is at least 95%. A subsample is 'good' if it consists of p good observations of the sample, which may contain (in the most extreme case) up to 50% of bad observations. The expression for this probability is

$$1 - (1 - (1/2)^p)^m \qquad \text{if } n/p \text{ is large.}$$

(This idea was already used by Stahel in 1981 for multivariate location.) When n and p are rather small, all possible combinations of p points out of n are considered instead of the repeated random subsamples.

The basic idea of this algorithm is illustrated in the artificial two-dimensional example below:



For this case n equals 9 and p equals 2. The algorithm will handle all pairs of points out of the 9. We will restrict the explanation for only three such combinations, namely (f,g), (f,h) and (g,h). Let us start with the points f and h. The regression surface (which is a line here) passing through the points f and h is found by solving the system of equations

$$y' = \theta_1^\circ x' + \theta_2^\circ$$

$$y'' = \theta_1^\circ x'' + \theta_2^\circ$$

where (x',y') and (x'',y'') are the coordinates of respectively f and h. The trial estimate θ_1° and θ_2° are the unknows. Then, the residuals $y_i - \theta_1^\circ x_i - \theta_2^\circ$ corresponding with this line are determined for each point i in the sample. The median of the squared residuals (which is the objective function) is calculated and compared with the best value eventually found for previous pairs of points. As a minimization of the squared residuals has to be performed, the trial estimate corresponding with the points f and h will be retained only when it leads to a lower objective function value. Examining the scatterplot above, it is easy to find out that the pair of points (f,g) will be the 'best' out of the three combinations considered. Indeed, the majority of the observations have a small residual with respect to the line passing through f and g. Repeating this procedure for each pair of points will finally yield the lowest objective function value.

When handling a regression model with intercept, the estimator of location is used for finding the constant term. Once $\hat{\theta}_1,\ldots,\hat{\theta}_{p-1}$ are found, $\hat{\theta}_p$ is the LMS estimate of location of the sample constituted by

$$z_i = y_i - x_{1,i}\hat{\theta}_1 - \ldots - x_{p-1,i}\hat{\theta}_{p-1}\ , \qquad i=1,\ldots,n.$$

Apart from the regression coefficients, the scale parameter σ (σ=standard deviation of the e_i) has to be estimated in a robust way. For that purpose a preliminary scale estimate s° is calculated. This s° is based on the value of the objective function, multiplied by a finite-sample correction factor (which depends on n and on p) for the case of normal errors:

$$s^\circ = [\min_{\hat{\theta}} \operatorname{median}_i r_i^2]^{1/2} \times 1.4826 \times (1 + 5/(n-p)) \qquad (2.2).$$

The factor $1.4826 = 1/\Phi^{-1}(3/4)$ was introduced because $\operatorname{med}_i |\xi_i| / \Phi^{-1}(3/4)$ is a consistent estimator of σ when the

random variables ξ_i are distributed like $N(0,\sigma)$. From an empirical study, it appeared that the scale estimator was too small in normal error situations, especially for small samples. Therefore the multiplication with the factor $1 + 5/(n-p)$, which has been derived from a simulation study, was necessary.

With this scale estimate the standardized residuals r_i/s° are computed and used to determine a weight w_i for the i-th observation as follows:

$$w_i \begin{cases} 1 & \text{if } |r_i/s^\circ| < 2.5 \\ 0 & \text{elsewhere} \end{cases} \qquad (2.3)$$

Then the final scale estimate for the LMS regression is given by

$$\sigma^* = [\ (\sum_{i=1}^{n} w_i\, r_i^2) \ / \ (\sum_{i=1}^{n} w_i - p)\ ]^{1/2}.$$

At the classical model, σ^* would be a consistent estimator of σ if the weights w_i are independent of the data (x_i, y_i).

This algorithm has been implemented in FORTRAN and runs on an IBM PC or compatible computer. We called it PROGRESS: program for robust regression. The output of PROGRESS consists of results concerning LS and concerning reweighted LS based on the LMS, which is described below. For both methods, PROGRESS gives the regression coefficients with their standard deviations and T-values, their variance-covariance matrix, an estimate for the scale parameter σ, the determination coefficient (R squared), the standardized residuals, and residual plots of two types. PROGRESS provides also two different options for handling data sets with missing values.

In order to have other classical regression results, like F-tests and options for variable selection, one could run PROGRESS first and then use the weights provided by the LMS in a standard package (for example BMDP or SAS). Pursuing

this course, one is safeguarded against outliers which may disturb the ordinary LS regression analysis.

A skilful study of the residuals is an important task of applied regression analysis. Therefore PROGRESS has a plot option which permits to obtain for both regression techniques a plot of the standardized residuals versus the estimated value of y, or a plot of the standardized residuals versus the index of the observation i (this is called an index plot). A point in the scattergram is represented by a digit. This digit corresponds to the number of points having approximately the same coordinates. When more than 9 points coincide, an asterisk '*' is printed on that position. In problems with several variables, the residual plots corresponding to the reweighted LS estimator are very useful for spotting the outlying observations. If the residual plot of both the robust and non-robust regression method agree closely, the LS result can be trusted.

In the residual plot a dotted line is drawn through zero and a horizontal band on the interval [-2.5,2.5] is marked. These lines facilitate the interpretation of the results. When the observed y_i value equals the estimated y_i value, then the resulting residual becomes zero. Points in the neighbourhood of this zero line are best fitted by the model.

If the residuals are normally distributed, then one can expect that roughly 98% of the standardized residuals will lie in the interval [-2.5,2.5]. In the residual plots of the reweighted LS, the outliers are far away from this zone. So observations for which the standardized residual is situated far from the horizontal confidence band can be identified as outlying. A warning must be given for this interpretation on the residual plots corresponding to the LS estimator. A true outlier does not necessary possess a large LS residual. The distortion produced by the outlier(s) pushes the otherwise 'good' observations away from the regression hyperplane. This effect makes it nearly impossible to identify the 'bad' observation(s). This phenomenon is also illustrated by the example in section 4.

Besides the identification of outliers, the residual plots contain also very important information for detecting common types of model inadequacies. A pattern showing that the variance of the residuals increases or decreases with increasing estimated y, points out that it could be favourable to apply a suitable transformation to either an explanatory variable or the response variable. A pattern resembling a horse-shoe may be caused by nonlinearity. In this case a transformation on an explanatory or on the response variable, or an additional squared term in the model, or the addition of another explanatory variable may be required.

3. ROBUST REGRESSION ESTIMATES DERIVED FROM THE LMS REGRESSION

Several methods exist for improving the efficiency of the LMS. Some of these are presented in this section.

3.1 The reweighted least squares regression

The reweighted least squares regression (RLS) technique consists of minimizing the sum of the squared residuals multiplied by a weight w_i

$$\underset{\hat{\theta}}{\text{minimize}} \sum_{i=1}^{n} w_i r_i^2 \qquad (3.1).$$

The weights w_i are determined from the LMS solution as in equation (2.3) but with the final scale estimate $\sigma*$ instead of s^o. In this way, the result is protected against the presence of outlying points by means of the weights based on the robust LMS estimator.

3.2 The one-step M-estimator

An M-estimate is defined as a solution $\theta = (\theta_1, \ldots, \theta_p)^t$ of the system of equations

$$\sum_{i=1}^{n} x_{ji}\, \psi(r_i / \hat{\sigma}) = 0$$

The function ψ is absolutely continuous with derivative ψ'.

We use the tangens-hyperbolicus function as defined by Hampel, Rousseeuw and Ronchetti (1981):

$$\Psi(x) = x \qquad \text{for } 0 < |x| < p$$
$$= (A(k-1))^{\frac{1}{2}}\tanh[\tfrac{1}{2}((k-1)B^2/A)^{\frac{1}{2}}(c-|x|)]\operatorname{sign}(x) \qquad \text{for } p < |x| < c$$
$$= 0 \qquad \text{for } c < |x|$$

where p=1.470089, c=3.0, k=5.0, A=.680593 and B=.769313.
Let $\theta^*=(\theta_1^*,\ldots,\theta_p^*)^t$ be the vector of an initial solution (we will take here the LMS estimates of $\theta=(\theta_1,\ldots,\theta_p)^t$ and σ^* the corresponding estimate for the scale parameter σ.)
Bickel (1975) defined a one-step M-estimate as

$$\hat{\theta} = \theta^* + \frac{\sigma^*}{B(\psi,\Phi)}(\psi(r_1^*/\sigma^*),\ldots,\psi(r_n^*/\sigma^*))X^t(XX^t)^{-1}$$

where $B(\psi,\Phi) = \int \psi'(u)d\Phi(u)$ and X is the pxn matrix containing the explanatory variables.

4. AN EXAMPLE

In order to illustrate the output provided by PROGRESS we have chosen for the famous 'stackloss data' set presented by Brownlee (1965). The data describe the operation of a plant for the oxidation of ammonia to nitric acid. The 3 explanatory variables and the response variable can be described as follows:

x_1 rate of operation
x_2 cooling water inlet temperature
x_3 acid concentration
y stack loss

We will use a linear regression model with constant term (this is obtained by creating a fourth explanatory variable which takes on the value one for all cases).

We have selected this example because it is a set of real data and it has been examined by a great number of statisticians (Draper and Smith (1966), Daniel and Wood (1971), Andrews (1974), Atkinson (1980) and many others) with the help of several methods. Summarizing their findings, it can be said that most people concluded that observations 1,3,4,21 were outliers. According to some people, observation 2 is reported as an outlier too. Running PROGRESS on this data set gives rise to the following output:

```
*************************************************************
* ROBUST MULTIPLE LINEAR REGRESSION WITH A CONSTANT.        *
*************************************************************

NUMBER OF CASES =    21
NUMBER OF COEFFICIENTS (INCLUDING CONSTANT TERM) =      4

THE EXTENSIVE SEARCH ALGORITHM WILL BE USED.
DATA SET = BROWNLEE STACK LOSS DATA
     THIS ROBUST MULTIPLE REGRESSION ALGORITHM IS BASED ON
     THE LEAST MEDIAN OF SQUARES (LMS) METHOD.
     (SEE P.ROUSSEEUW (1984),LEAST MEDIAN OF SQUARES REGRESSION,
     JOURNAL OF THE AMERICAN STATISTICAL ASSOCIATION,79,871-880)
     THIS PROGRAM HAS BEEN WRITTEN BY A.LEROY AND P.ROUSSEEUW.
     FOR FURTHER INFORMATION OR COMMENTS, PLEASE CONTACT
                    A.LEROY
                    VRIJE UNIVERSITEIT BRUSSEL
                    C.S.O.D. (M203)
                    FLEINLAAN 2
                    B-1050 BRUSSELS (BELGIUM)

PRINT OPTION =  2
PLOT OPTION  =  2
THERE ARE NO MISSING VALUES.

YOUR DATA RESIDE ON FILE        : B:GSTACK.DAT
THE OBSERVATIONS

      OPERATION  TEMPERATUR ACID CONC.  STACKLOSS
 1      80.0000     27.0000     89.0000     42.0000
 2      80.0000     27.0000     88.0000     37.0000
 3      75.0000     25.0000     90.0000     37.0000
 4      62.0000     24.0000     87.0000     28.0000
 5      62.0000     22.0000     87.0000     18.0000
 6      62.0000     23.0000     87.0000     18.0000
 7      62.0000     24.0000     93.0000     19.0000
 8      62.0000     24.0000     93.0000     20.0000
 9      58.0000     23.0000     87.0000     15.0000
10      58.0000     18.0000     80.0000     14.0000
11      58.0000     18.0000     89.0000     14.0000
12      58.0000     17.0000     88.0000     13.0000
13      58.0000     18.0000     82.0000     11.0000
14      58.0000     19.0000     93.0000     12.0000
15      50.0000     18.0000     89.0000      8.0000
16      50.0000     18.0000     86.0000      7.0000
17      50.0000     19.0000     72.0000      8.0000
18      50.0000     19.0000     79.0000      8.0000
19      50.0000     20.0000     80.0000      9.0000
20      56.0000     20.0000     82.0000     15.0000
21      70.0000     20.0000     91.0000     15.0000

MEDIANS  =

      OPERATION  TEMPERATUR ACID CONC.  STACKLOSS
        58.0000     20.0000     87.0000     15.0000

DISPERSIONS =

      OPERATION  TEMPERATUR ACID CONC.  STACKLOSS
         5.9304      2.9652      4.4478      5.9304

THE STANDARDIZED OBSERVATIONS

      OPERATION  TEMPERATUR ACID CONC.  STACKLOSS
 1      3.7097      2.3607      .4497      4.5528
 2      3.7097      2.3607      .2248      3.7097
 3      2.8666      1.6862      .6745      3.7097
 4       .6745      1.3490      .0000      2.1921
 5       .6745       .6745      .0000       .5059
 6       .6745      1.0117      .0000       .5059
 7       .6745      1.3490     1.3490       .6745
 8       .6745      1.3490     1.3490       .8431
 9       .0000      1.0117      .0000       .0000
10       .0000      -.6745    -1.5738      -.1686
11       .0000      -.6745      .4497      -.1686
12       .0000     -1.0117      .2248      -.3372
13       .0000      -.6745    -1.1242      -.6745
14       .0000      -.3372     1.3490      -.5059
15     -1.3490      -.6745      .4497     -1.1804
16     -1.3490      -.6745     -.2248     -1.3490
17     -1.3490      -.3372    -3.3725     -1.1804
18     -1.3490      -.3372    -1.7986     -1.1804
19     -1.3490       .0000    -1.5738     -1.0117
20      -.3372       .0000    -1.1242       .0000
21      2.0235       .0000      .8993       .0000

SPEARMAN RANK CORRELATION COEFFICIENTS BETWEEN THE VARIABLES
( STACKLOSS  IS THE OUTPUT VARIABLE)

OPERATION     1.00
TEMPERATUR     .74  1.00
ACID CONC.     .61   .36  1.00
STACKLOSS      .92   .85   .50  1.00

PEARSON CORRELATION COEFFICIENTS BETWEEN THE VARIABLES
( STACKLOSS  IS THE OUTPUT VARIABLE)

OPERATION     1.00
TEMPERATUR     .78  1.00
ACID CONC.     .50   .39  1.00
STACKLOSS      .92   .88   .40  1.00

*************************************************************
```

LEAST SQUARES REGRESSION
****

VARIABLE	COEFFICIENT	STAND. ERROR	T - VALUE
OPERATION	.71564	.13486	5.30661
TEMPERATUR	1.29529	.36802	3.51957
ACID CONC.	-.15212	.15629	-.97331
CONSTANT	-39.91968	11.89600	-3.35572

SUM OF SQUARES = 178.83000

SCALE ESTIMATE = 3.24336

VARIANCE - COVARIANCE MATRIX =

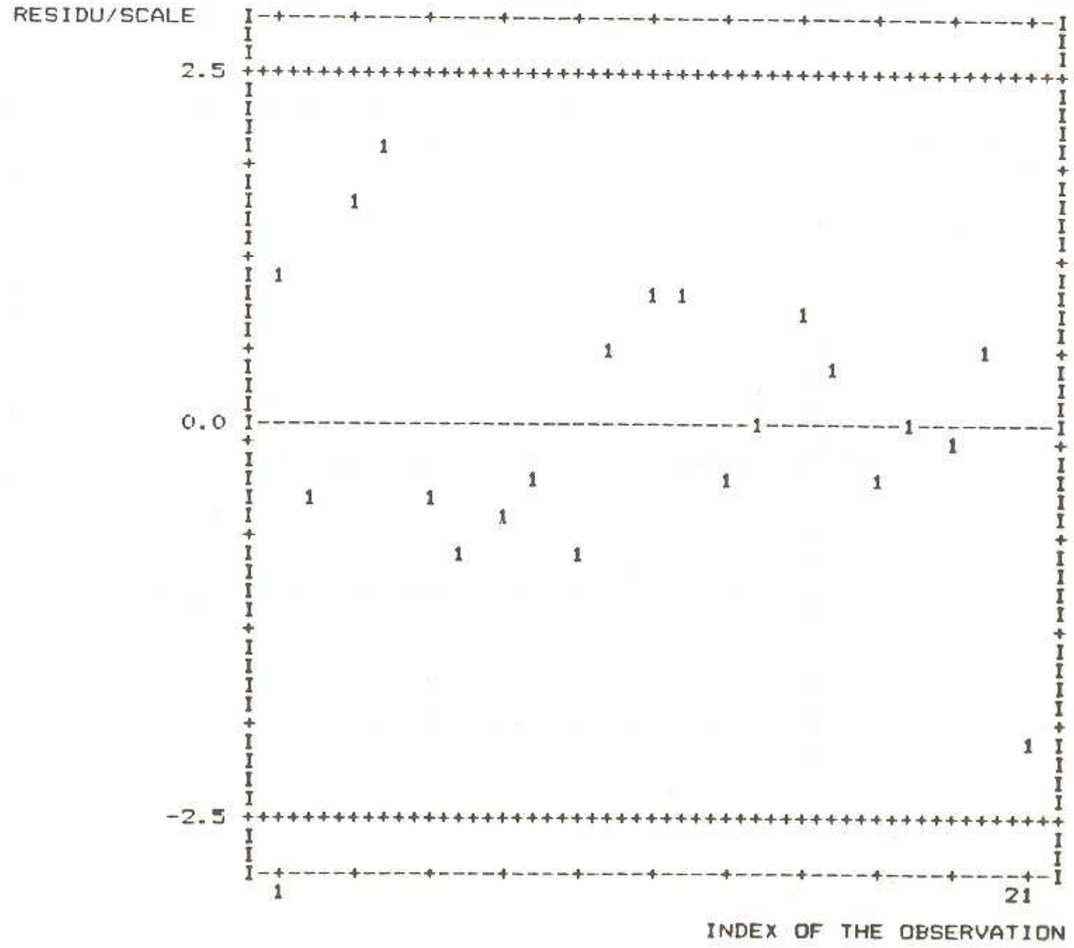
.1819D-01			
-.3651D-01	.1354D+00		
-.7144D-02	.1048D-04	.2443D-01	
.2876D+00	-.6518D+00	-.1676D+01	.1415D+03

COEFFICIENT OF DETERMINATION (R SQUARED) = .91358

OBSERVED STACKLOSS	ESTIMATED STACKLOSS	RESIDUAL	NO	RES/SC
42.00000	38.76536	3.23464	1	1.00
37.00000	38.91748	-1.91748	2	-.59
37.00000	32.44447	4.55553	3	1.40
28.00000	22.30222	5.69778	4	1.76
18.00000	19.71165	-1.71165	5	-.53
18.00000	21.00694	-3.00694	6	-.93
19.00000	21.38949	-2.38949	7	-.74
20.00000	21.38949	-1.38949	8	-.43
15.00000	18.14438	-3.14438	9	-.97
14.00000	12.73280	1.26720	10	.39
14.00000	11.36370	2.63630	11	.81
13.00000	10.22054	2.77946	12	.86
11.00000	12.42856	-1.42856	13	-.44
12.00000	12.05050	-.05050	14	-.02
8.00000	5.63858	2.36142	15	.73
7.00000	6.09495	.90505	16	.28
8.00000	9.51995	-1.51995	17	-.47
8.00000	8.45509	-.45509	18	-.14
9.00000	9.59826	-.59826	19	-.18
15.00000	13.58785	1.41215	20	.44
15.00000	22.23771	-7.23771	21	-2.23

BROWNLEE STACK LOSS DATA

--- L E A S T S Q U A R E S ---



****

LEAST MEDIAN OF SQUARES REGRESSION
****

THE MINIMIZATION OF THE 12TH QUANTILE OF THE SQUARED RESIDUALS IS PERFORMED.

ON A TOTAL OF 2092 SUBSAMPLES (OF 4 POINTS OUT OF 21)
92 SUBSAMPLES LED TO A SINGULAR SYSTEM OF EQUATIONS.
THE SOLUTION IS ONLY BASED ON THE GOOD SAMPLES.

MULTIPLE LMS SOLUTION
****

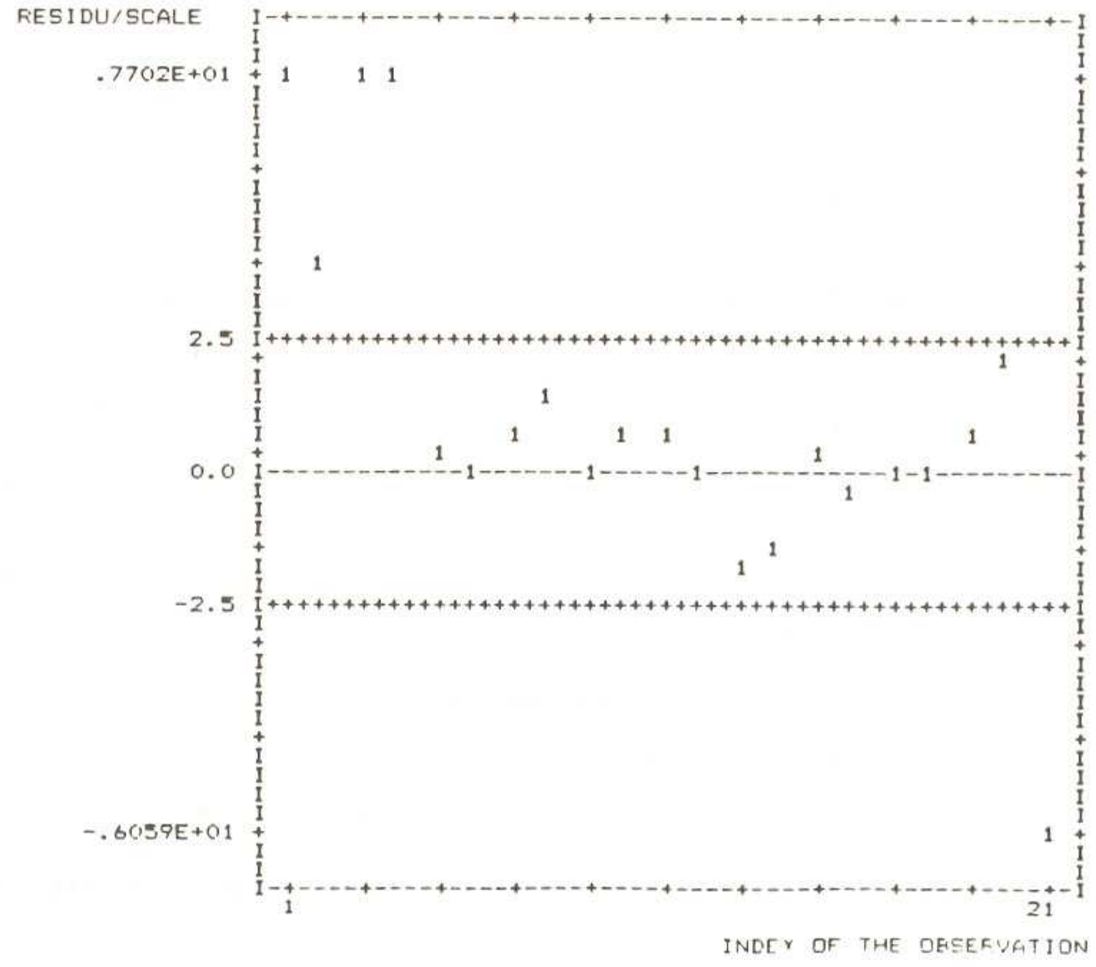
VARIABLE	COEFFICIENT
OPERATION	.71429
TEMPERATUR	.35714
ACID CONC.	.00000
CONSTANT	-34.50000

FINAL SCALE ESTIMATE = 1.26134

COEFFICIENT OF DETERMINATION = .97105

OBSERVED STACKLOSS	ESTIMATED STACKLOSS	RESIDUAL	NO	RES/SC
42.00000	32.28572	9.71428	1	7.70
37.00000	32.28572	4.71428	2	3.74
37.00000	28.00000	9.00000	3	7.14
28.00000	18.35714	9.64286	4	7.64
18.00000	17.64286	.35714	5	.28
18.00000	18.00000	.00000	6	.00
19.00000	18.35714	.64286	7	.51
20.00000	18.35714	1.64286	8	1.30
15.00000	15.14286	-.14286	9	-.11
14.00000	13.35714	.64286	10	.51
14.00000	13.35714	.64286	11	.51
13.00000	13.00000	.00000	12	.00
11.00000	13.35714	-2.35714	13	-1.87
12.00000	13.71429	-1.71429	14	-1.36
8.00000	7.64286	.35714	15	.28
7.00000	7.64286	-.64286	16	-.51
8.00000	8.00000	.00000	17	.00
8.00000	8.00000	.00000	18	.00
9.00000	8.35714	.64286	19	.51
15.00000	12.64286	2.35714	20	1.87
15.00000	22.64286	-7.64286	21	-6.06

BROWNLEE STACK LOSS DATA



REWEIGHTED LEAST SQUARES BASED ON THE LMS
****

VARIABLE	COEFFICIENT	STAND. ERROR	T - VALUE
OPERATION	.68609	.07358	9.32433
TEMPERATUR	.56710	.12872	4.40576
ACID CONC.	-.01725	.05305	-.32519
CONSTANT	-35.48420	3.80302	-9.33053

WEIGHTED SUM OF SQUARES = 16.02457

CORRESPONDING SCALE ESTIMATE = 1.11229

VARIANCE - COVARIANCE MATRIX =

.5414D-02			
-.4512D-02	.1657D-01		
-.1917D-02	-.3882D-03	.2814D-02	
-.5118D-01	-.4384D-01	-.1246D+00	.1446D+02

COEFFICIENT OF DETERMINATION (R SQUARED) = .96288

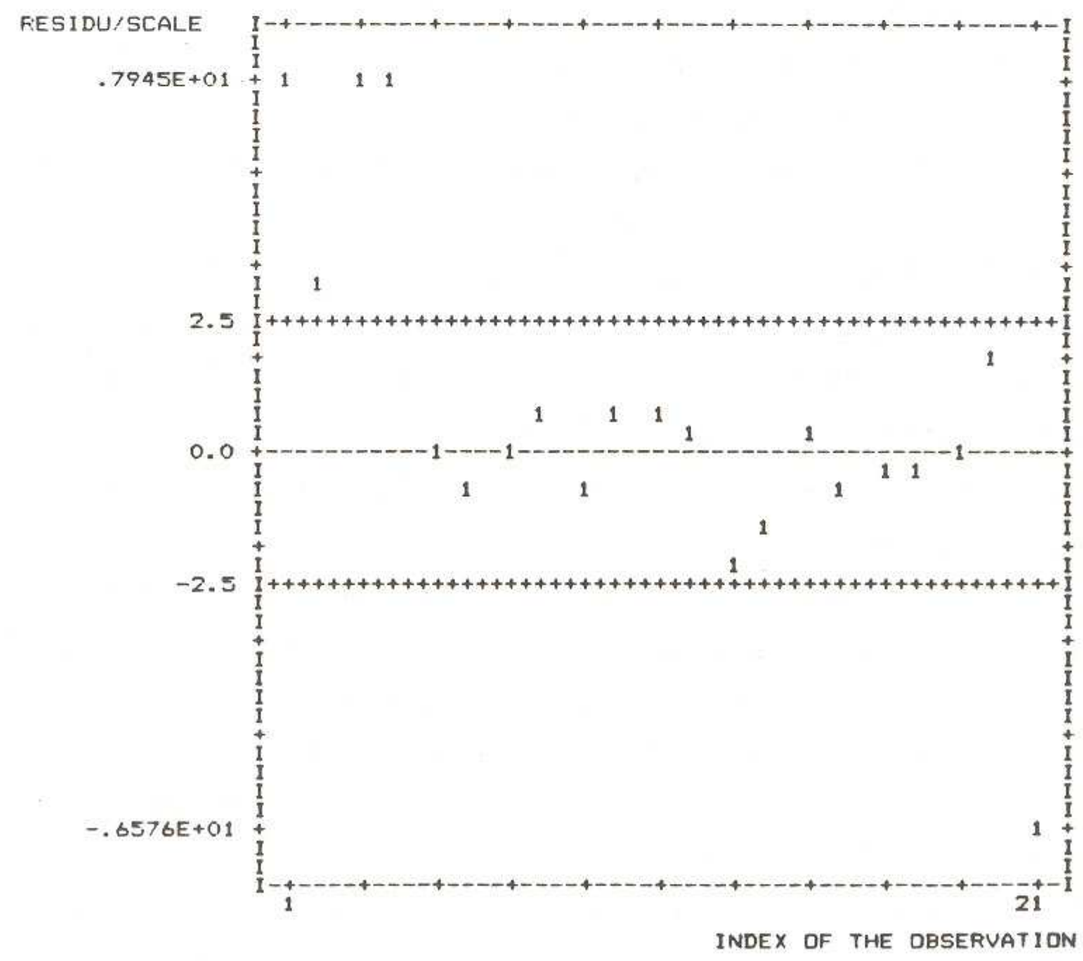
THERE ARE 16 POINTS WITH NON-ZERO WEIGHT.

AVERAGE WEIGHT = .76190

OBSERVED STACKLOSS	ESTIMATED STACKLOSS	RESIDUAL	NO	RES/SC	WEIGHT
42.00000	33.17970	8.82030	1	7.93	.0
37.00000	33.19695	3.80305	2	3.42	.0
37.00000	28.59778	8.40222	3	7.55	.0
28.00000	19.16323	8.83677	4	7.94	.0
18.00000	18.02902	-.02902	5	-.03	1.0
18.00000	18.59612	-.59612	6	-.54	1.0
19.00000	19.05973	-.05973	7	-.05	1.0
20.00000	19.05973	.94027	8	.85	1.0
15.00000	15.85175	-.85175	9	-.77	1.0
14.00000	13.13700	.86300	10	.78	1.0
14.00000	12.98174	1.01826	11	.92	1.0
13.00000	12.43189	.56811	12	.51	1.0
11.00000	13.10249	-2.10249	13	-1.89	1.0
12.00000	13.47984	-1.47984	14	-1.33	1.0
8.00000	7.49300	.50700	15	.46	1.0
7.00000	7.54475	-.54475	16	-.49	1.0
8.00000	8.35336	-.35336	17	-.32	1.0
8.00000	8.23260	-.23260	18	-.21	1.0
9.00000	8.78246	.21754	19	.20	1.0
15.00000	12.86451	2.13549	20	1.92	1.0
15.00000	22.31456	-7.31456	21	-6.58	.0

BROWNLEE STACK LOSS DATA

--- R E W E I G H T E D L S (B A S E D O N L M S) ---



Examining the residual plot of the reweighted LS confirms that the observations 1,3,4 and 21 are outliers, as their residuals lie far from the confidence band. Observation number 2 is an intermediate case because it is just on the verge of the area containing the outliers. However, the residual plot corresponding to the LS fit masks the bad points.

Concluding this example we would like to emphasize that it is necessary to compare the standardized residuals of both the LS and the robust method in each regression analysis. Only the robust technique can be used as a reliable tool for diagnosing the outliers.

REFERENCES

ANDREWS, D.F. (1974), "A Robust Method for Multiple Linear Regression," Technometrics,16,523-531.

ATKINSON, A.C. (1980), "Examples showing the use of two graphical displays for the detection of influential and outlying observation in regression," Compstat 1980,276-282.

BELSLEY, D.A., KUH, E., and WELSH, R.E. (1980), "Regression Diagnostics," Wiley, New York.

BICKEL, P.J. (1975), "One-Step Huber Estimates in the Linear Model," J.A.S.A.,70,428-434.

BROWNLEE, K.A. (1965), "Statistical Theory and Methodology in Science and Engineering," John Wiley, 2nd ed., New York.

COOK, R.D. and WEISBERG, S. (1982), Residuals and Influence in Regression," Chapman and Hall, London.

DANIEL, C. and WOOD, F.S. (1971), "Fitting Equations to Data," John Wiley, New York.

DONOHO, D.L. and HUBER, P.J. (1983), "The Notion of Breakdown Point," in A Festschrift for Erich Lehmann, edited by P.Bickel, K.Doksum, and J.L.Hodges,Jr, Wadsworth International Group, Belmont, California.

DRAPER, N.R. and SMITH, H. (1966), "Applied Regression Analysis," John Wiley, New York.

HAMPEL, F.R. (1971), "A General Qualitative Definition of Robustness," Annals of Mathematical Statistics,

42, 1887-1896.

HAMPEL, F.R. (1975), "Beyond Location Parameters: Robust Concepts and Methods," Bulletin of the International Statistical Institute, 46, 375-382.

HAMPEL, F.R., ROUSSEEUW, P.J. and RONCHETTI, E. (1981), "The Change-of-Variance Curve and Optimal Redescending M-Estimators," Journal of the American Statistical Association, 76, 643-648.

LEROY, A. and ROUSSEEUW, P.J. (1984), "PROGRESS: A Program for Robust Regression," Research Report no. 201, Centrum voor Statistiek en O.R., University of Brussels.

ROUSSEEUW, P.J. (1984), "Least Median of Squares Regression," Journal of the American Statistical Association, 79, 871-880.