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A METHOD TO LOCATE THE MAXIMUM CIRCLE(S) INSCRIBED IN A POLYGON

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ABSTRACT

The problem introduced in this paper regards the location of an obnoxious facility at the maximum distance from the surrounding urban areas. In particular, if these areas lie close to each other, along the perimeter of a suitable polygon (convex or not), then the notion of a "rolling circle" (which is the basic idea behind the algorithm proposed) can be used, in order to locate the maximum circle inscribed in that polygon. Its center will be then the appropriate position for the location of the obnoxious facility.

1. *Introduction*

The problem that will be tackled in this paper is that of locating the maximum circle inscribed in a polygon (convex or not). It belongs to a class of locational problems termed "maximin". This class of problems has a wide range of applications, mainly in the location of obnoxious facilities -facilities dangerous for the man and his environment.

Assuming that the perimeter of a given polygon represents the frontier of some geographical area (an urban area or an environmentally sensitive one) extending outside the polygon and requiring "protection", we seek to position the obnoxious facility inside the polygon, as far as possible from the nearest point of the perimeter.

More realisticly, some remote area may be examined, for the location of the obnoxious facility (see figure 1). We consider this area to be "confined" by a number of urban districts lying around it. Representing, then, those districts by suitable polygons and joining them together with appropriately chosen straight line segments (beams), we can enclose the remote area inside a (closed) polygonal line, consisted by beams and district perimeter sections, in an alternating sequence.

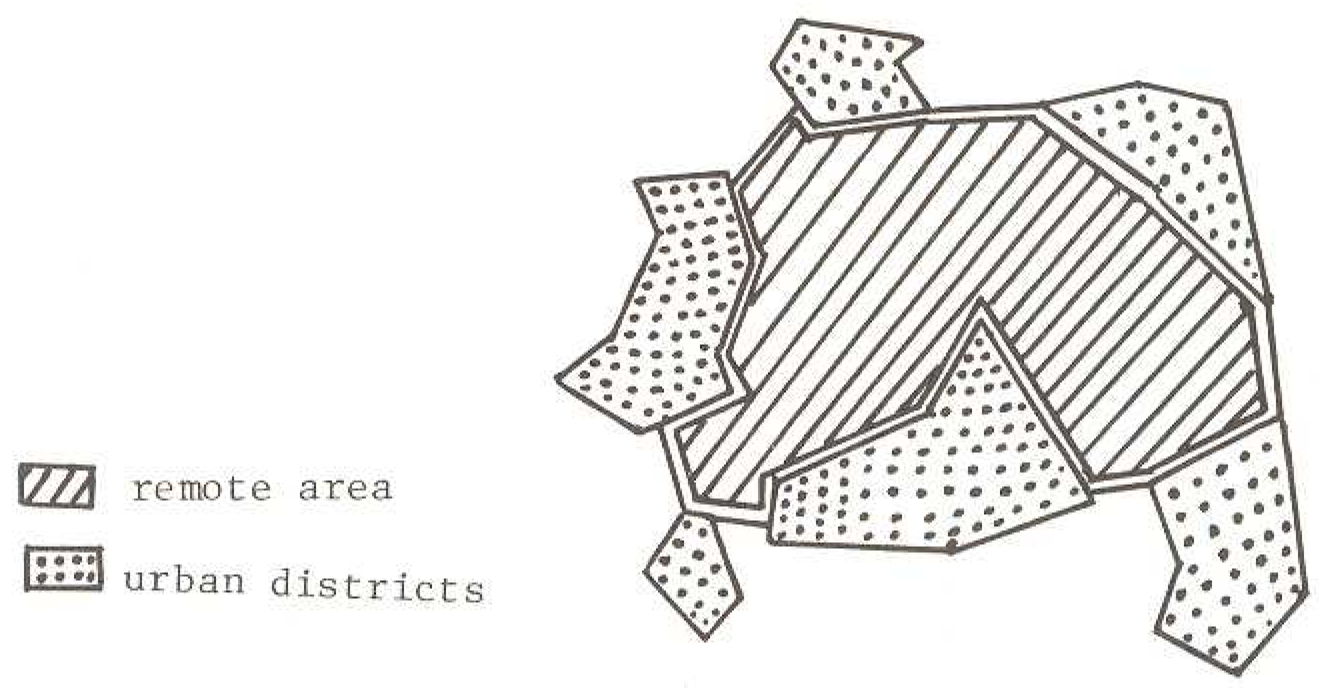

Figure 1. Remote Area and Urban Districts.

One can extend the above problem in the space as follows. A polyhedron is considered (some polyhedral warehouse) and the maximum sphere, inscribed in it, is to be located (we may plan to store some dangerous substance, eg. radioactive substance, and ask for a central location inside the warehouse, as distant as possible, from its bounding surface).

The algorithm proposed here -we call it NONVEX (NON CONVEX)- can be easily extended to solve the above problem in the space.

What follows in section 2, is an introduction to the basic ideas behind the method proposed, ie the notion of the Rolling Circle, of Base and Pilot, Barrier and Active Area. In section 3, we develop the theoretical background of the method, whereas in section 4 an algorithmic presentation of the method is given. Finally, in section 5, a generalization of the problem is proposed, while in section 6, the order of the algorithm proposed is examined.

2. *Introduction to the Algorithm*

2.1. *List of Notations*

n : the number of vertices of the polygon

x(P), y(P): the x and y coordinates of a point P

$\mathcal{P}$: the polygon

$\mathcal{P}_0$: the perimeter of the polygon

$\mathcal{V} = \{V_1, V_2, \ldots, V_n\}$: the anti-clockwise ordered set of vertices of the polygon, where V_1 is the "westmost" vertex, ie $V_1 \in \mathcal{V}: x(V_1) \leqq x(V)$ for any $V \in \mathcal{V}$

$\mathcal{S} = \{s_1, s_2, \ldots, s_n\}$: the anti-clockwise ordered set of sides of the polygon, where s_1 is the side joining vertices V_1 and V_2

$\mathcal{C} = \mathcal{C}(C, r)$: the circle with center at point C and radius r

$\mathcal{C}_0 = \mathcal{C}_0(C, r)$: the circumference of the circle $\mathcal{C}(C, r)$

$\overline{P_1P_2}$ denotes the straight line defined by points P_1 and P_2

$\overline{P_1P_2}$ denotes the line segment confined by points P_1 and P_2

$\overleftarrow{P_1P_2}$ denotes the half-line origined at point P_1 and passing through point P_2

$d(P_1,P_2)$ the Euclidean distance, from point P_1 to P_2

2.2 *The Notion of the "Rolling Circle"*

Consider a point P_t, that scans the perimeter of the polygon, starting from vertex V_1 and moving anti-clockwise, until it reaches V_1 again -P_t will be called thereon Base Tangential Point, or BTP for short.

Associate with BTP a circle C_t, which is the maximum one among the circles inscribed in the polygon and touching its perimeter at BTP (in case BTP coincides with a non-convex vertex, the direction of the vector with origin at BTP and end at C_t must be given). C_t will be called thereon Rolling Circle,, or RC for short. Since BTP is moving in a continuous manner, the trajectory of the RC center is itself a continuous line passing, obviously, through all points at which the optimal circles are centered. As we will prove later, the trajectory of the RC center is composed by a series of straight- line and parabola segments. The essential idea, in this paper, is to follow the trajectory of the RC center, in order to locate the center(s) of the maximum circle(s) inscribed in the polygon.

The main problems encountered with this approach were:

(a) to produce a suitable method of search, overcoming the continuous nature of the trajectory of the RC center -due to the continuous movement of BTP.

(b) to find a way to calculate the parameters involved in the equations describing the various segments of this trajectory.

The proposed method offers efficient and simple answers to the above problems. We start with the first problem, that is the method of search. We will show that, limiting appropriately and in a piece-wise manner the movement of BTP along P_0, we can decompose the trajectory of the RC center into simpler pieces, that are either straight-line segments or parts of a parabola. Both the above types of trajectory segments can be described through parameters associated with some simple and well studied equations. This decomposition, based

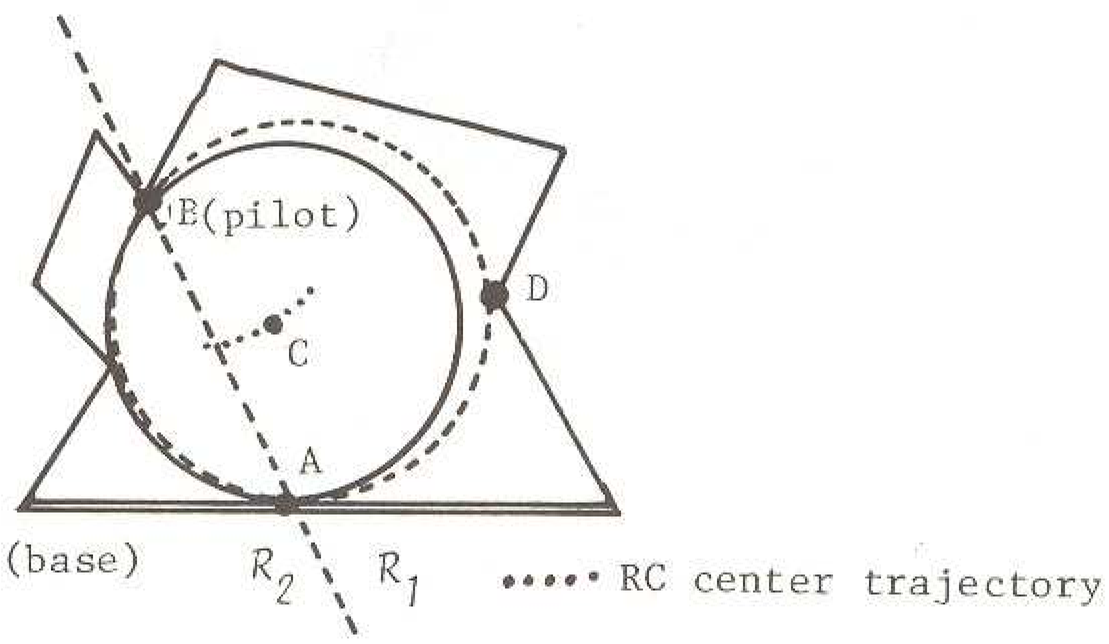

Figure 2. Tangential Points of the Rolling Circle.

on the a-priori knowledge of the trajectory segment parameters (that can be calculated at the beginning of each segment) allows jumps for BTP -and hence for the RC center- from any "local" optimal position to the next. Consequently, in order to transform the continuous search into a discrete one, we only have to make BTP visit those points of the polygon perimeter, which correspond to the turning points of the RC center trajectory (there is only a finite number of such turning points).

Let A be the current position of BTP, in its anti-clockwise movement, and C(C,r) the associated RC. According to lemma 4 (section 3), the circle C is tanget to P_0 not only at A, but also at some other points -at least one. Let B be the one immediately next to A, in an anti-clockwise direction along C_0 (see figure 2). Call this point Pilot Tangential Point, or PTP for short. Note that the direction of movement of PTP is clockwise along P_0, as BTP keeps moving anti-clockwise. That brings PTP and BTP closer and closer to each other, until they coincide at some convex vertex of the polygon (see figure 3).

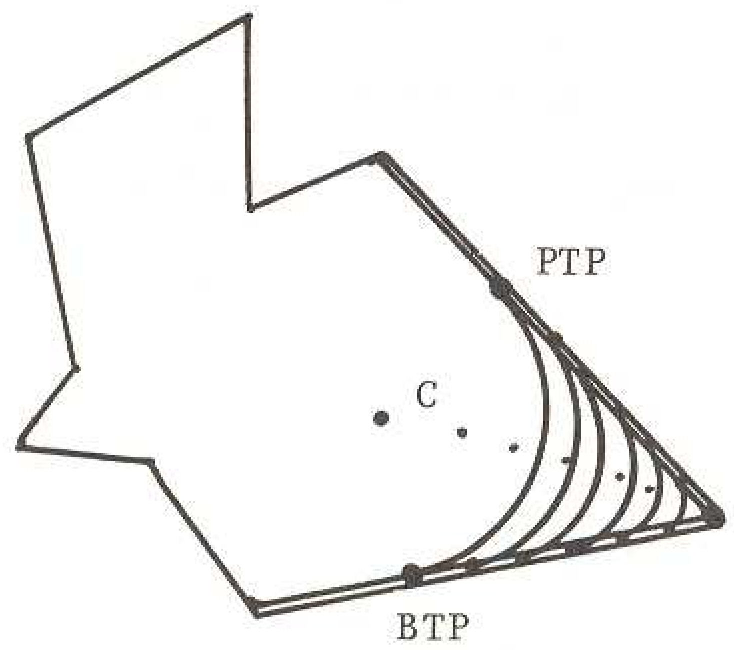

Figure 3. BTP and PTP Converging to a Vertex.

At this point, we introduce two of our basic concepts, namely "base" and "pilot". We call "base" the polygon side where BTP belongs, or, in case BTP coincides with a non-convex vertex, we restrict the notion of "base" to that very vertex (we intend to show later, that the characteristics of the base, as one of the two trajectory modulators -the other is the pilot- change dramaticly when the base changes from side to vertex or vice-versa).

Consider, next, the straight line $\overline{AB}$ and the part of P_0 belonging to the sub-plane R_1 (see figure 2). Note that R_1 is the sub-plane confined by $\overline{AB}$ and lying to the right of the vector AB (in the direction that BTP is about to move). It is obvious that C shares no other tangential points with $P_1 = P_0 \cap R_1$ except A and B, whereas it may touche $P_2 = P_0 - P_1$, at one ore more other points.

As the RC current position changes, its "rolling" is obstructed by no points of P_2. Actually, all these points -including the currently tangential ones-will be left outside RC, as soon as the position of the RC center changes (in case A is interior to a polygon side, this happens when BTP leaves A, in order to continue its anti-clockwise movement, along P_0). Then, the only tangential points (for a while) will be BTP and PTP, lying on the current base and pilot respectively. Hence, the RC will be led in its move by the base and pilot, for as long as, moving on its current course, it can remain inscribed in the polygon. The critical moment is when other polygon "elements" (sides or vertices), located between the current base and pilot, become tangential to the so called "RC front" (the part of the RC circumference, from BTP to PTP). Note that both BTP and PTP are considered as belonging to the RC front.

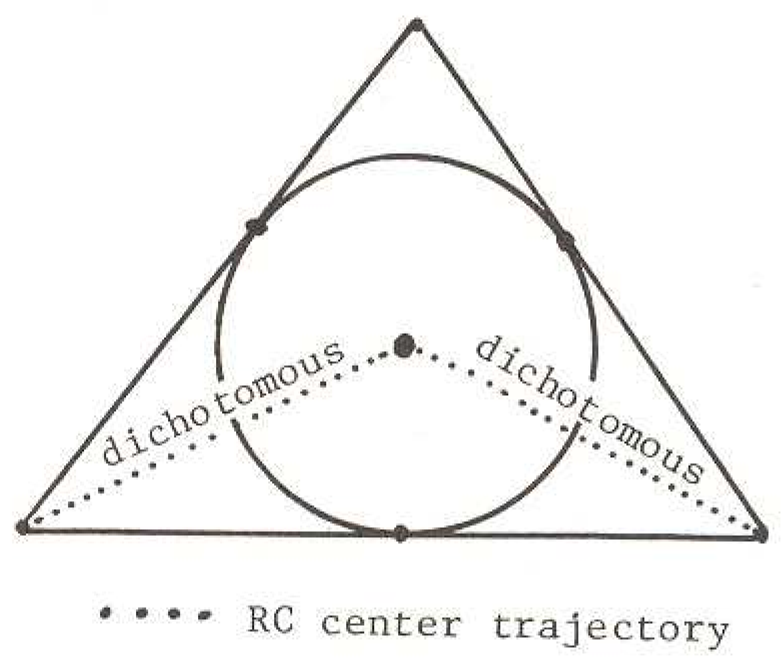

Figure 4a. A Piece-Wise Linear RC Center Trajectory.

We use the term "tangential element", when referring, either to a vertex of P lying on the RC circumference, or to a polygon side being tangent to RC. Note that, it is possible for both a polygon side and one of its adjoint vertices, to be, simultaneously, tangential elements (when, in fact, RC is tangent to a side, at one of its adjoint vertices). Whenever any third element, besides the current base and pilot, colides with the moving front, the course of the trajectory of the RC center is altered. For this, all such elements are called "barriers".

The point on the trajectory of the RC center, at which a change of the trajectory type (or of parameters) occurs, is called "turning point". To be more specific, at a turning point, the type of trajectory may remain unaltered, whereas its parameters change (as in figure 4a -from/to straight line, only slope changes), or else, the type itself is being changed (as is in figure 4b -from straight line to parabola- at A, and vice-versa at B).

The type and parameters of the trajectory, between two turning points (or between the old position of the center, at C, and the new position, at the time of collision of the RC front with the barrier D -see figure 2) is determined exclusively by the base and pilot. On the other hand, the turning points of the trajectory are determined by the type and position of the barriers (a barrier may be a vertex or an interior point of a side).

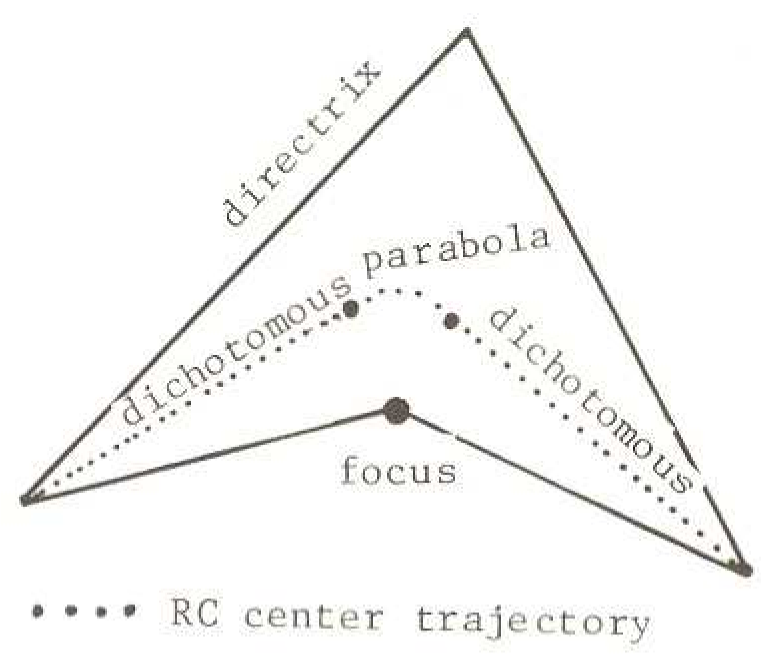

Figure 4b. RC Center Trajectory Composed by Straight Line and Parabola Segments.

We will study next, more analyticly, the basic for the algorithm notions of base, pilot, and barrier.

2.3. *Determining the Type of Trajectory Between Two Turning Points (the Role of Base and Pilot).*

The possible combinations for the types of base and pilot (as hosters of BTP and PTP correspondingly) are:

(i) base is a side and pilot is a side

(ii) base is a side and pilot is a non-convex vertex

(iii) base is a non-convex vertex and pilot is a side

(iv) base and pilot are both non-convex vertices.

In case (i) (see figure 5a), the center of RC is free to move on the (interior) dichotomous of the angle formed by the base and pilot sides. While, in case (ii) (see figure 5b), the center is free to move on the parabola focused at PTP (the pilot) with the base forming the directrix. In case (iii)(see figure 5c), there is an infinitude of circles inscribed in the polygon and tangent to it, at point BTP (which, in this case, is a non-convex vertex). Mo-

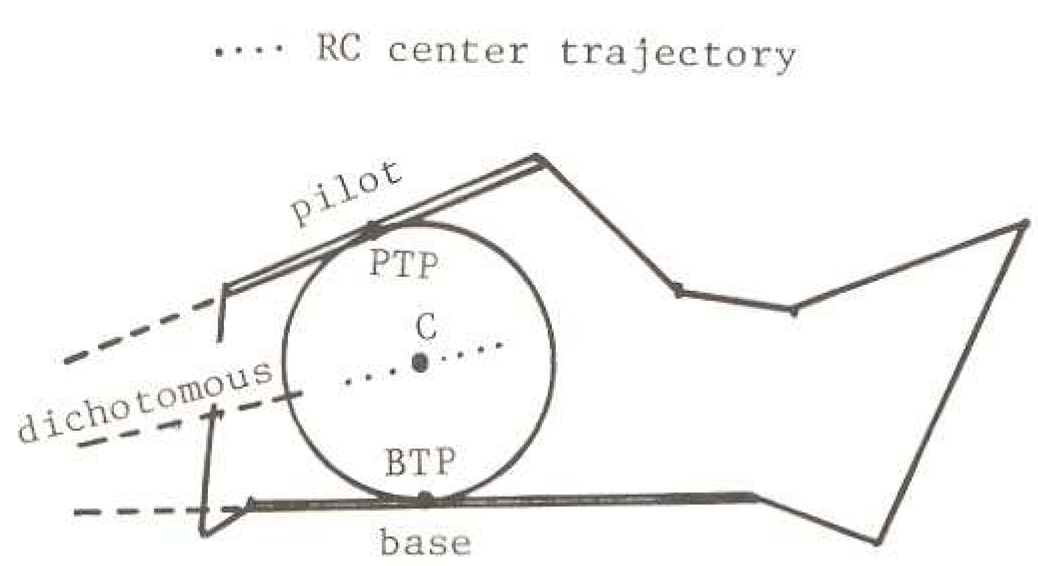

Figure 5a. Base and Pilot Edges: Linear Trajectory.

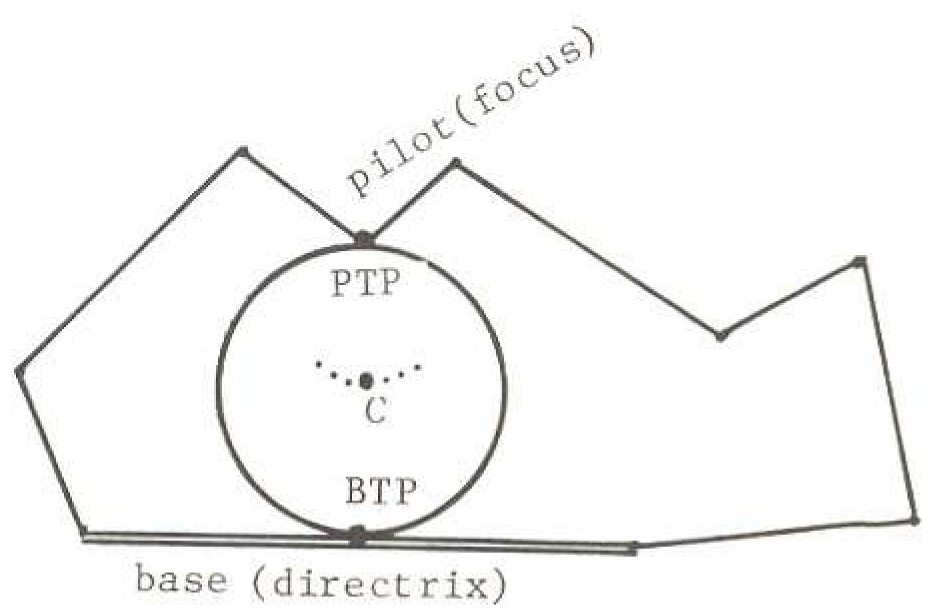

Figure 5b. Base Edge, Pilot Non-convex Vertex:Parabolic Trajectory.

re specificly, for each direction, in the angle formed by the normals to the sides converging to BTP, there is a single maximum inscribed circle, tangent to

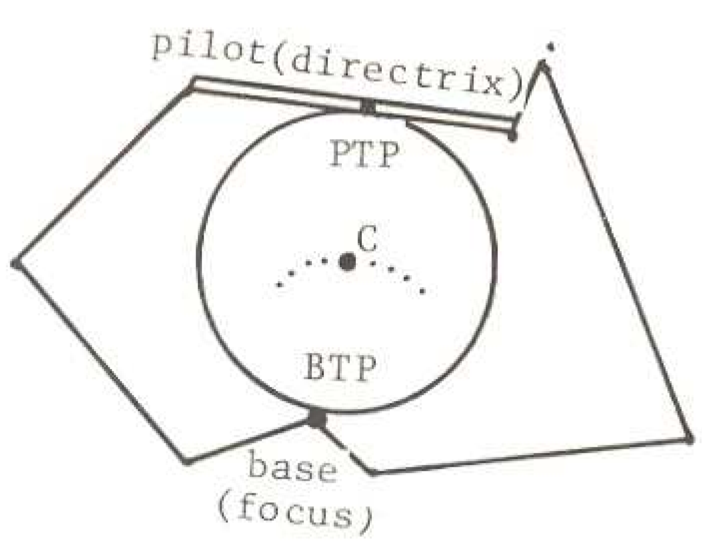

Figure 5c. Base Nonconvex Vertex, pilot Edge:Parabolic Trajectory.

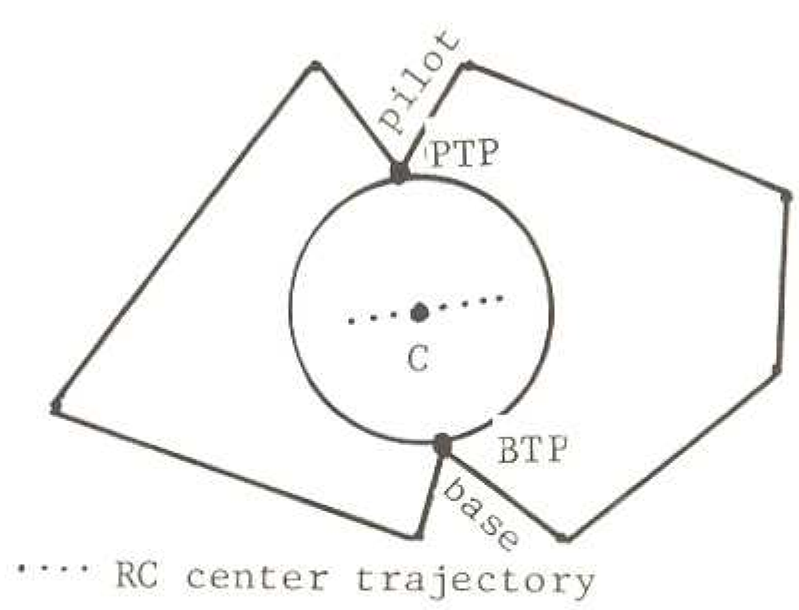

Figure 5d. Base and Pilot Nonconvex Vertices:Linear Trajectory.

P_0 at BTP, with its center lying in this direction. Thus, in case (iii), the RC can hardly be regarded as "rolling", since it is rather "turning" round a steady point of its circumference, namely the non-convex vertex hosting BTP. As a consequence, there is here -as in case (ii)- only one degree of freedom in the RC movement, the one that confines the course of the RC center, on the parabola focused at BTP and having as directrix the pilot side. Finally, in case (iv) (see figure 5d), where base and pilot are both non-convex vertices , the RC center moves on the mid-perpendicular of the straight line segment confined by BTP and PTP.

2.4. *Determining the Turning Points of the Trajectory (the Role of Barriers)*

The types of trajectory described in the previous section, depend only on the elements of base and pilot. Any third element lying on the path joining BTP and PTP (in the anti-clockwise direction) represents a "candidate barrier" for RC, which, in the absence of other barriers, would force the trajectory to change at some point -call it "candidate turning point". If C is the current turning point on the trajectory, then the "candidate barier"that is associated with the nearest to C (along the trajectory) candidate turning point, is the barrier actually forcing the trajectory to turn -we call it "actual barrier".

After the collision of the RC front with the actual barrier, the latter (if different from the element right adjoint to the base) will automaticly replace the pilot and keep this place until a new barrier is encountered. The role of the actual barrier as the new pilot gives a hint of how to locate the turning points on the trajectory of the RC center.

Suppose that the "current trajectory" of the RC center -ie the trajectory corresponding to the current base and pilot- has already been located, according to the methodology described in the previous section. Now, for each candidate barrier, regarded as taking the place of the current pilot (while the base remains the same), a new trajectory is obtained -by repetitive application of the above method. The intersection of the current trajectory with each new trajectory (corresponding to some candidate barrier) is a "candidate turning point", whose distance, along the current trajectory, from the last turning point, can be easily compared to the distance of the rest. Then, the location of the nearest candidate turning point -which is, actually, the next turning point- is a trivial process. The following example refers to the process of locating the next turning point.

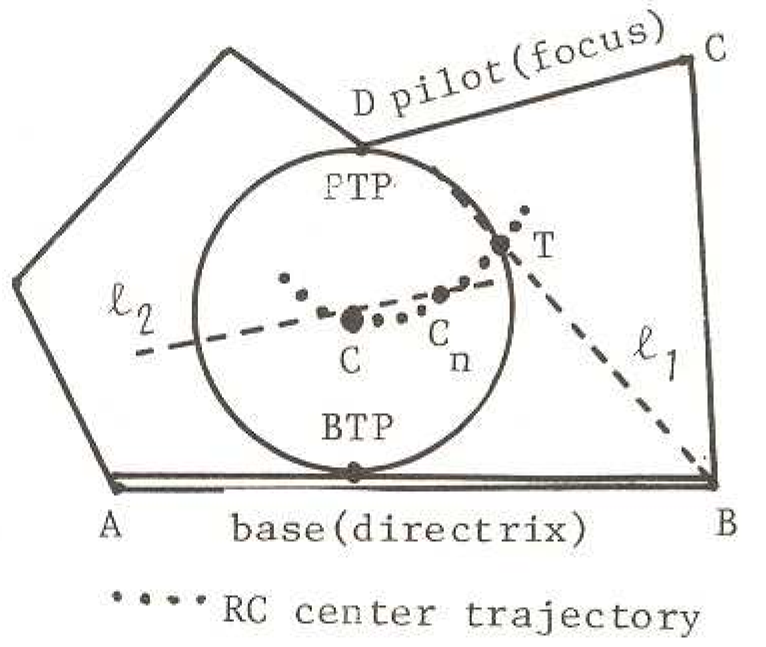

Figure 6. The Next Turning Point.

Current base : side AB

Current pilot : non-convex vertex D

Current trajectory: the part of the parabola "to the right" of the current turning point C

Candidate barriers: sides BC and CD

Side BC : The trajectory corresponding to AB, as base, and BC, as pilot, is the dichotomous $\ell 1$ of the angle formed by the sides BC and AB. Associated candidate turning point:point T

Side CD : The trajectory corresponding to AB, as base, and BC, as pilot, is the dichotomous $\ell 2$ of the angle formed by the sides CD and AB. Associated candidate turning point:point C_n

Point C_n : the next turning point -as being the nearest to C

Side CD : the associated "actual barrier"

2.5. *Notion of the Active Area.*

Recall that, in order to calculate the candidate turning point corresponding to the Current Candidate Barrier -call it CCB for short-, we disregard all elements except the base, the pilot, and the specific CCB, leaving the base and the pilot to modulate the course starting from the current RC position -denote by C_c this "Current Circle" - and CCB, alone, to confine it.

The RC, at the moment of collision with CCB, defines a "Limiting Circle" denoted by C_L. Moreover, the part of the C_L circumference, which lies to the right of BTP and to the left of PTP, is called C_L front -since its definition is analogous to that of the C_c front. The area, then, confined by the C_c front, the base, the C_L front, and pilot, is called "Active area" and is denoted by A_a (see figure 7). In other words, A_a is the area scanned by the RC front, in its way from the position of the C_c front, to that of the C_L front.

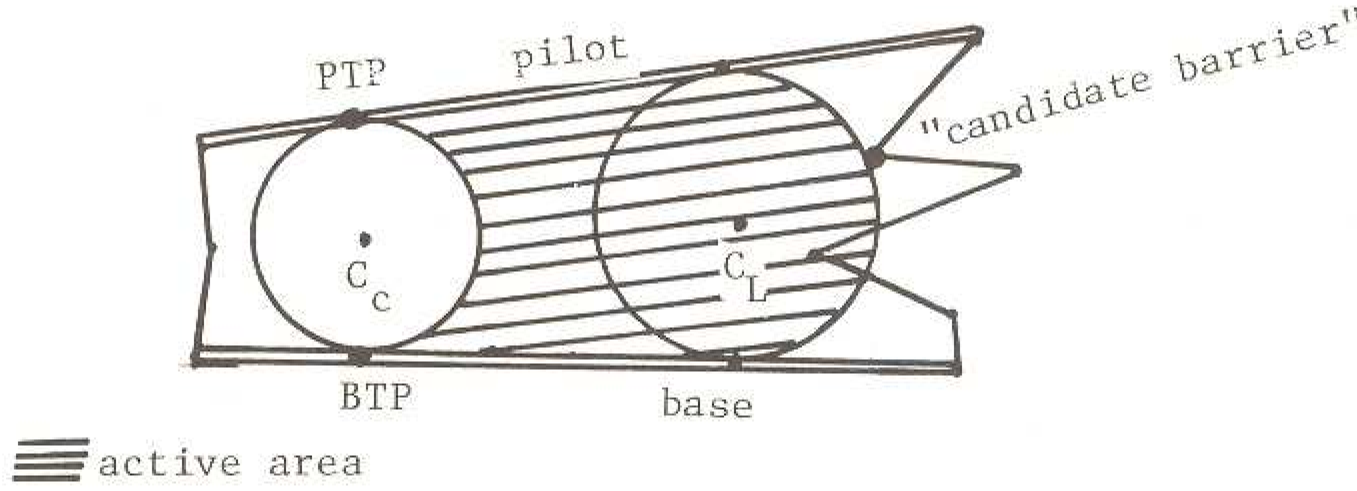

Figure 7. The Active Area.

Since the specific CCB determining A_a is encountered by the moving RC front, before any candidate barrier lying outside A_a, the latter are all excluded from further consideration. On the contrary, any candidate barrier lying inside A_a is encountered before CCB, by the RC front. Thus, a new C_L front is determined, by that barrier, and a smaller active area is obtained, destined to replace the current A_a.

As for the algorithm proposed, in order to locate the actual barrier (among all candidate ones), we itteratively reduce A_a, each time using, as CCB, the first candidate barrier found inside the current A_a. The iterative scheme terminates when all candidate barriers have been checked. It is reminded that, as candidate barriers, are considered all elements of the polygon (sides and vertices) forming the part of the perimeter lying "to the right" of the base, between the base and the pilot.

3. *Some Theoretical Results*

Lemma 1. Any maximum inscribed, in a polygon P, circle -thereon denoted by MIC- is tangential to P_o, at two points at least.

Proof

We first prove that any MIC is tangent to P_o, at one point at least. Assuming that it shares no (tangential) points with P_o, we will be led to a contradiction.

If C(C,r) is the MIC under consideration (figure 8) and P is the closest to C point of P_o then, by assumption, $r^*=d(P,C)>r$. Then circle $C^*(C,r^*)$ is inscribed in P and clearly greater than C(C,r). Hence the latter cannot be a MIC.

We prove, next, that a MIC, say C(C,r), cannot be tangent to P_o at exactly one point. Again, assuming that it has one exactly tangential point, say P, will result to contradiction.

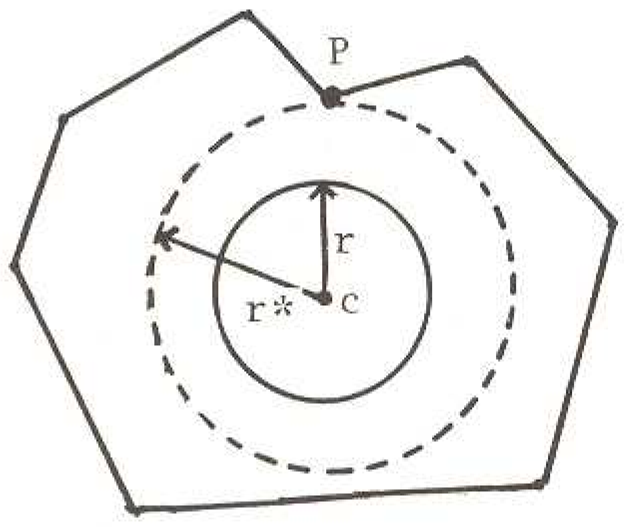

Figure 8. Lemma 1.

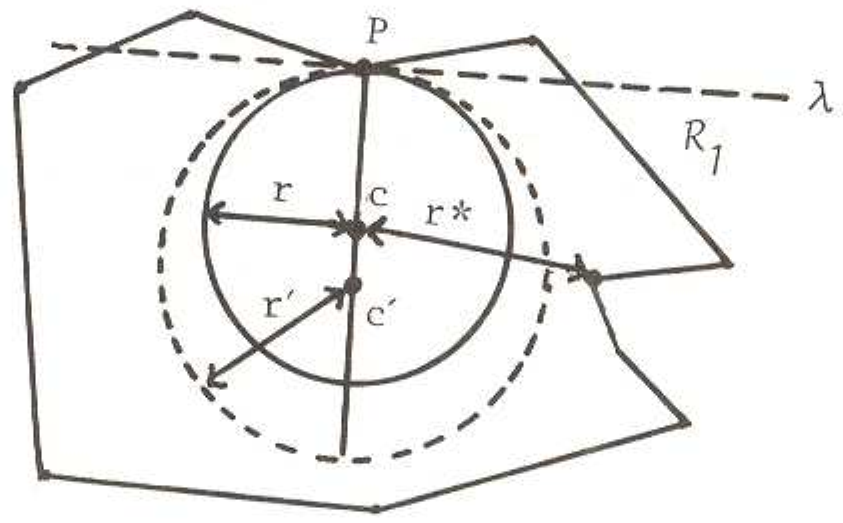

Figure 9. Lemma 1.

Let λ be the straight line tangent to C at point P (figure 9), while R_1 is the (open) sub-plane defined by λ and the center C. We define $P_1=P_0\cap R_1$ and $P_2=P_0-P_1$. We will prove that there exists a circle $C'(C',r')$, inscribed in the polygon, with $r'>r$. In order to do this, take

$$r^*=\min\{d(C,x):x\in P_1\} \tag{1}$$

and

$$\epsilon^*=r^*-r. \tag{2}$$

Then

$$\epsilon^*>0, \tag{3}$$

since, by assumption, C has no common points with P_1.

Let $C'(C',r')$ be a circle that lies on R_1, is tangent to λ at P, and has radius $r'=r+(\epsilon^*/2)$. Clearly, C' is greater than C. As we will prove, it is also inscribed in the polygon, or, equivalently, $d(C',x')\geqq r$ for any $x\in P_0$. It is evident that

$$d(C',x)>d(C',P) \tag{4}$$

for any $x\in P_2$

If $x\in P_1$ (figure 9) then, by (1), (2),

$$d(C,x)\geqq r+\epsilon^* \tag{5}$$

But $r+\epsilon^*=r+(\epsilon^*/2)+(\epsilon^*/2)=r'+(\epsilon^*/2)$ and, hence, (5) is equivalent to

$$d(C,x)\geq r'+(\epsilon^*/2) \tag{6}$$

Considering, now, the triangle $CC'x$, we get

$$d(C,x)\leq d(C',x)+d(C,C') \tag{7}$$

Relations (6) and (7) give

$$d(C',x)+d(C,C')\geq r'+(\epsilon^*/2)$$

or

$$d(C',x)\geq r'+(\epsilon^*/2)-d(C,C') \tag{8}$$

But, by the definition of r',

$$d(C,C')=r'-r=r+(\epsilon^*/2)-r=\epsilon^*/2$$

and, thus, (8) can be written as

$$d(C',x)\geq r'+(\epsilon^*/2)-(\epsilon^*/2)=r' \tag{9}$$

for any $x\in P_1$

Relations (4) and (9) give

$$d(C',x)\geq r' \tag{10}$$

for any $x\in P_0$

or, equivalently, circle $C'(C',r')$ is inscribed in the polygon. Furthermore, C' is greater than C. This contradicts the fact that C is a MIC.

Lemma 2. If $C(C,r)$ is MIC for a polygon P and shares exacty two (tangential) point with P_0, then these points are anti-diametrical and belong, as interior points, to two parallel sides.

Proof

We prove, first, that the two tangential points, say P_1 and P_2, are anti-diametrical. Assuming that they are not, we will be led to a contradiction.

If P_1 and P_2 are not anti-diametrical, then the lines ℓ_1, ℓ_2, tangent to C at P_1, P_2, respectively, are not parallel and, hence, they intersect each other, say at a point A (figure 10). If now $P^*=P\cap\Gamma$, where $\Gamma=\widehat{P_1AP_2}$, then C is inscribed in P^* (since C is inscribed in both P and Γ) and is tangent to the perimeter of P^*, at exactly two points, namely P_1 and P_2.

Furthermore, if s^*_1 and s^*_2 are the sides of P^*, where P_1 and P_2 lie respectively, then it is obvious that P_1 and P_2 are interior points of these sides (since the vertices adjoint to sides s^*_1 and s^*_2 are convex).

The conclusions, thereon, are related to the following idea: The circle C can be "inflated" while remaining inscribed in P^* and tangent to sides s^*_1 and s^*_2 , which will be called thereon "base" and "pilot" respectively. Because its center is free to slide on the dichotomous δ of C, until the inflating circumference of Γ touches the part of P_0, say P_1, confined by the lines ℓ_1 and ℓ_2 and located opposite to A, with respect to circle C (note that s^*_1 and s^*_2 don't belong to P_1).

In mathematical terms, we define:

$$P_1=P_0\cap\Gamma\cap R_1, \tag{1}$$

where R_1 is the subplane confined by $\overline{P_1P_2}$ and containing C, and

$$P_2=P_0\cap\Gamma\cap R_2$$

where R_2 is the complement of R_1. During its "inflation", the circle will remain inscribed in P^* and, hence, in P_0. This contradicts the initial assumption, that C is a MIC.

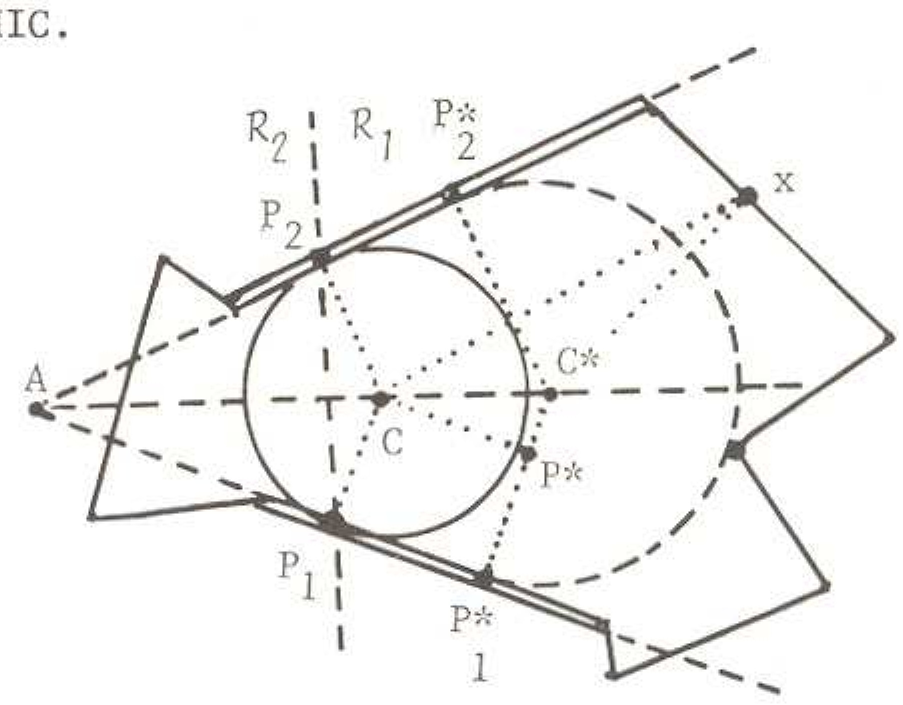

Figure 10. Lemma 2.

More analyticly, if r^* is chosen as follows

$$r^* = \min\{d(C,x) : x \in P_1\}, \tag{2}$$

then

$$\epsilon^* = r^* - r > 0, \tag{3}$$

since C has no common points with P_1. From relations (2) and (3)

$$d(C,x) \geqq \epsilon^* + r \tag{4}$$

for any $x \in P_1$. Let us consider, next, the circle $C^*(C^*, r^*)$, centered on the dichotomous δ, at a distance $\epsilon^*/2$ from C, to the direction of inflation. It is immediate that $r^* = d(C^*, P_1^*) = d(C^*, P^*) + d(P^*, P_1^*) = d(C, C^*)\ \sin\ (\Gamma/2) + r$, where P_1^* and P_2^* are the tangential points of C with ℓ_1 and ℓ_2, correspondingly, and P^* the projection of C on $\widetilde{C^*P_1^*}$. Hence

$$r^* = (\epsilon^*/2)\ \sin\ (\Gamma/2) + r \tag{5}$$

On the other hand, if $x \in P_1$, then from triangle CC^*x

$$d(C^*,x) > d(C,x) - d(C,C^*) = d(C,x) - (\epsilon^*/2),$$

and because of (4)

$$d(C^*,x) > \epsilon^* + r - (\epsilon^*/2) + r. \tag{6}$$

But

$$(\epsilon^*/2) + r = r^* + (\epsilon^*/2) \cdot \sin\ (\Gamma/2) + (\epsilon^*/2) \cdot (1 - \sin\ (\Gamma/2))$$

and, due to (5),

$$(\epsilon^*/2) + r = r^* + (\epsilon^*/2) \cdot (1 - \sin\ (\Gamma/2)) \tag{7}$$

Now, (6) and (7) result in

$$d(C^*,x) > r^* + (\epsilon^*/2) \cdot (1 - \sin(\Gamma/2)) > r^*$$

since $\epsilon^* > 0$ and $1 - \sin(\Gamma/2) > 0$. Consequently

$$d(C^*,x) > r^* \tag{8}$$

for any $x \in P_1$. Furthermore, it is obvious that

$$d(C^*,x) > r^* \tag{9}$$

for any $x \in P_2$. Considering, finally, the fact that C^* is inscribed in angle Γ, relations (8) and (9) result in

$$d(C^*,x)\geq r^*$$

for any $x\in P^*$ and, thus, in

$$d(C^*,x)\geq r^*$$

for any $x\in P_0$, which means that C^* is inscribed in P. But then, our initial assumption that C^* is a MIC, is contradicted by $r^*>r$ and, hence, the tangential points P_1 and P_2 are anti-diametrical points of C_0.

We will prove, next, that P_1 and P_2 are interior points of two (parallel) sides of P. It is enough to show that neither, of the two points, is a vertex.

To begin with, it is clear that P_1 and P_2 cannot be convex vertices. Suppose then, that one of them, say P_1, is a non-convex vertex. Based on this hypothesis, we will end up with a contradiction.

Let R_1 be the open sub-plane lying right to $\overline{P_1P_2}$ and R_2 its complement (where $\overline{P_1P_2}$ belongs). Consider the straight lines, ℓ_1 and ℓ_2, tanget to C at points P_1 and P_2, respectively (figure 11). Since P_1 and P_2 are anti-diametrical points (as we have already proved), lines ℓ_1 and ℓ_2 are parallel, defining a zone, say Z.

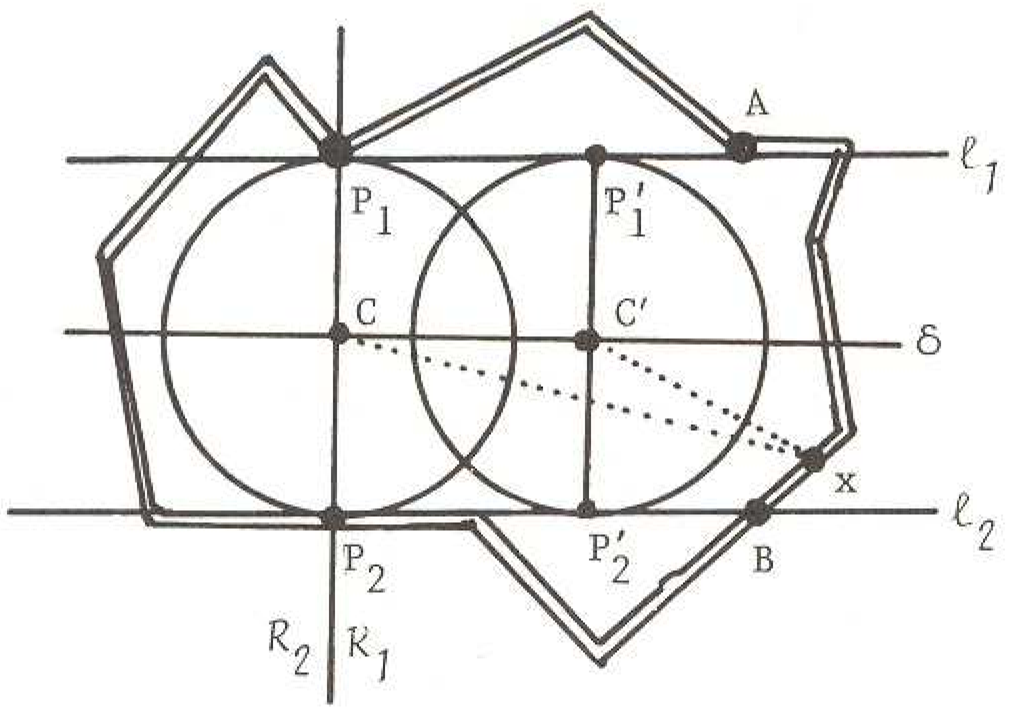

Figure 11. Lemma 2.

Let s_1^*, s_2^* be the edges of $P^*=P_0 \cap Z$, to which P_1, P_2 respectively belong and P_1, P_2 the parts of P^* as defined in (1). We, also, define A as the nearest to P_1 point to $s_1^* \cap P_0 \cap R_1$ and B as the right adjoint vertex of s_2^*. Then the following are evident.

$$r_1=d(P_1,A)>0 \tag{10}$$

(because we assumed that P_1 is a non-convex vertex)

$$d(C,x)>r$$

for any $x \in P_1$ (since $C(C,r)$ does not possess common points with P_1)

Hence

$$r_2=\min\{d(C,x):x \in P_1\}>0 \tag{11}$$

Also,

$$r_3=d(P_2,B)>0 \tag{12}$$

(because P_2 is an interior point of edge s_2^*)

Consider, next, the circle $C'(C',r)$ with center on the mid-perpendicular of $\overline{P_1P_2}$, at a distance ϵ^* to the right of C, where

$$\epsilon^*=\min\{r_1,r_2,r_3\}/2. \tag{13}$$

Due to (10), (11), and (12)

$$\epsilon^*>0 \tag{14}$$

If, now, P_1^* and P_2^* are the tangential points of C' with ℓ_1 and ℓ_2, respectively, then P_1^* is an interior point of $\overline{P_1A}$, because

$$d(P_1,P_1^*)=d(C,C')=\epsilon^*<r_1=d(P_1,A)$$

Hence and by the definition of A,

$$P_1^* \notin P_0 \tag{15}$$

Besides, it is obvious that C' lies strictly to the right of P_2, so

$$d(C',x)>r \tag{16}$$

for any $x \in P_2$

Next, we will prove that C' lies strictly to the left of P_1 (ie not even touching P_1). That is, $d(C',x)>r$ for any $x \in P_1$.

Indeed, if $x \in P_1$ then, from triangle $CC'x$, we get

$$d(C',x)>d(C,x)-d(C,C')=d(C,x)-\sigma^* \qquad (17)$$

Also, if we replace r^* with r_3 in (4), then we have

$$d(C,x)\geqq r_3+r \qquad (18)$$

for any $x \in P_1$

Relations (17) and (18) give

$$d(C',x)>r+r_3-\sigma^* \qquad (19)$$

But from (13) it is immediate that $r_3>\sigma^*$, or, $r_3-\sigma^*>0$, hence (19) results in

$$d(C',x)>r \qquad (20)$$

for any $x \in P_1$

From (15), (16), and (20), it can be concluded that C' possesses at one point, tangential to the perimeter of the polygon (if $P_2' \in P_0$ then P_2' is the one, otherwise there is none). On the other hand, C' has radius r and, thus, it must be a MIC. But, according to lemma 1, a MIC shares at least two tangential points with P_0. By this contradiction, we conclude that none of the tangential points P_1 and P_2 can be a non-convex vertex. This completes the proof of lemma 2.

Lemma 3. If a MIC of a polygon, say $C(C,r)$, has two common (tangential) points, P_1 and P_2, with the perimeter P_0 of a polygon P, then, for this polygon:

(i) there exists an infinite number of MIC having, each one of them, exactly two common (tangential) points with P_0 and

(ii) there exist at least two MIC, each having at least three common (tangential) points with P_0.

Proof

By lemma 1, points P_1 and P_2 are anti-diametrical and, hence, the straight lines, tangent to C at these points correspondingly, ℓ_1 and ℓ_2 are parallel. Let Z be the zone they define, $P^{*}=P_0 \cap Z$, and $P_3=s_1^{*} \cup s_2^{*}$, where s_1^{*} and s_2^{*} are the parallel edges of P^{*} on which P_1 and P_2 (as defined by (1) in lemma 2) lie respectively (figure 12). Then, clearly, $(P^{*}=P_1 \cup P_2 \cup P_3)$.

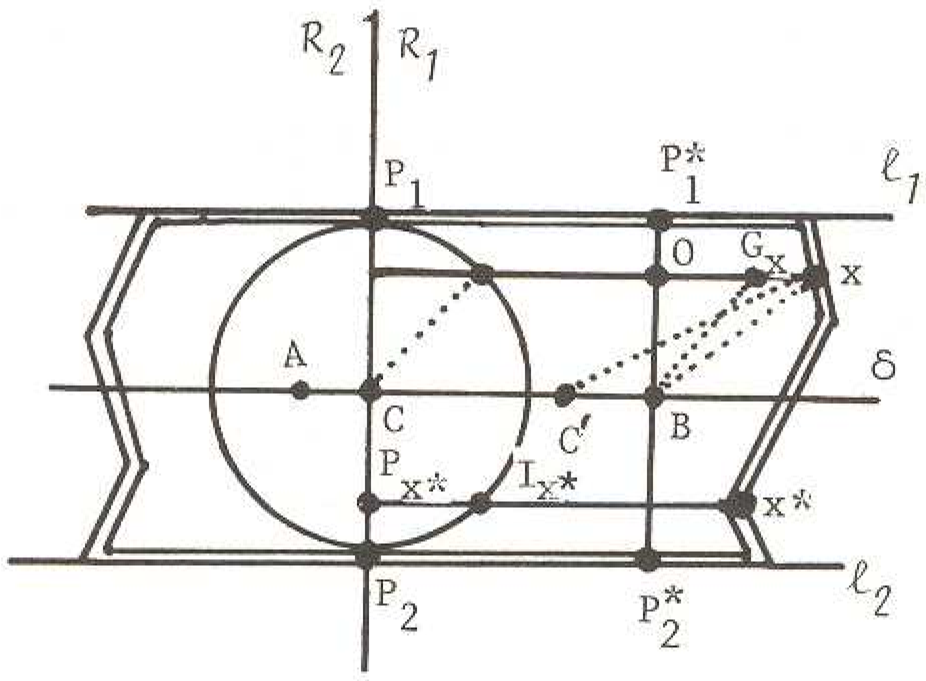

Figure 12. Lemma 3.

We have already proved (lemma 2) that C shares no common points with P_1 and P_2, whereas it has two common (tangential) points with P_3 (points P_1 and P_2). It is obvious (see the figure) that if C is forced to "roll" towards P_1 (or, alternatively, towards P_2) then, before it touches P_1 (or P_2), its center will have traced a non-zero line segment, on the mid-perpendicular δ of $\overline{P_1P_2}$, say $\overline{CB}$ (or $\overline{AC}$ -if moving towards P_2). Consequently, there is an infinite number of MIC, each having exactly two tangential points (hence (i) holds). All those circles have radius r and center on the open interval $\overline{AB}$.

In particular, at any one of the ends of this line segment, say B (or A), the "rolling circle", besides the two common points with P_3, will have at least one common point with P_1 (or P_2), and that will prove (ii).

To prove, actually, that these observations are correct, we consider the geometrical points P_1 and P_2, in the case where the circle is "rolling" towards P_1. Whatever the conclusions may be, for this case, will be also valid, in an immediate analogy, for the "rolling" towards P_2.

To begin with, it is obvious that

$$d(C',x) \geqq r \tag{1}$$

for any $C' \in \overleftarrow{BC}$, $x \in P_1$

Let P_x be the projection of x on $\overline{P_1P_2}$ and $I_x = C_0 \cap \overline{xP}_x$. We define, then, function $f(x)=d(x,I_x)$. It is clear, that the point of P_1, say x^*, which will be first touched by the circumference of the "rolling" circle, is that point of P_1 which minimises the function $f(x)$. Furthermore, $f(x^*)>0$, because $C(C,r)$ has no common points with P_1.

What is left now, is to determine the location of the center, call it B, at the instance the "rolling" circle touches x^*. Indeed, at that instance, $d(B,x^*)=r$ and hence B is the intersection of $C_0(x^*,r)$ with the bisector δ. Besides, it is clear that $CI_x x^*B$ is a parallelogram and consequently

$$d(C,B)=d(I_{x^*}x^*)=f(x^*)>0$$

If, now, P^*_1 and P^*_2 are the tangential points of $C^B(B,r)$ with ℓ_1 and ℓ_2 respectively, then P_1 lies to the right of $\overline{P^*_1P^*_2}$.
(because $d(x,P_x) \geqq d(x,I_x) \geqq d(x^*,I_{x^*})=d(C,B)=d(P_1,P^*_1)$ for any $x \in P_1$)

Consider, next, a circle $C'(C',r)$ with center $C' \in \overleftrightarrow{CB}$ (there are infinitely many such circles). Due to (1) C' lies strictly to the right of P_2 (it possesses no points of P_2). We will also prove, that it lies strictly to the left of P_1.

For any $x \in P_1$, consider point $G_x \in \overleftrightarrow{I_x x}$, such that $d(I_x,G_x)=f(x^*)$. It is obvious, that both G_x and x lie to the right of point $O=\overline{P^*_1P^*_2} \cap \overline{P_x x}$ and, also, that

$$d(O,G_x) \leqq d(O,X)$$

Hence

$$d(B,G_x) \leqq d(B,x) \tag{2}$$

But

$$d(C',x)>d(B,x) \tag{3}$$

Relations (2) and (3) result in

$$d(C',x)>d(B,G_x) \tag{4}$$

Now, since lines ℓ_1, ℓ_2, and δ are parallel while $(I_x,G_x)=d(I^*_x,G_{x^*})$ = $d(C,B)$, CBG_xI_x turns out to be a parallelogram, and thus

$$d(B,G_x)=d(C,I_x)=r \tag{5}$$

Finally, (4) and (5) give

$$d(C',x)>r \tag{6}$$

for any $x \in P_1$, that is, C' lies strictly to the left of P_1 and, hence, it has exaclty two common points with the P_0 (which are the two tangential points of P_3).

To finish, if we consider the circle $C^B(B,r)$, then it is evident that (6) becomes $d(B,x) \geqq r$ for any $x \in P_1$, because $d(B,x^*)=r$. Hence, C^B lies strictly to the right of P_2, and also to the left of P_1, while sharing at least one tangential point with P_1. Consequently, it shares at least three (tangential) points with the perimeter P_0 of the polygon.

Theorem 1. For any polygon, there is a MIC that has at least three common (tangential) points with the perimeter of this polygon.

Proof

Direct consequence of lemma 1 and lemma 3.

Lemma 4. The maximum circle $C(C,r)$, inscribed in a polygon P and tangent to P_0 at a point A, possesses -besides A- at least one more tangential point (in common with P_0).

Proof

Assuming that there are no other tangetial points besides A (figure 13)

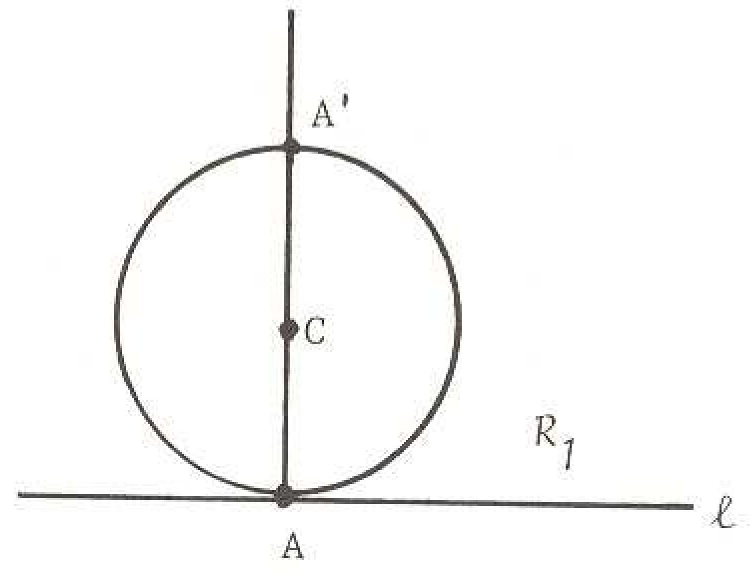

Figure 13. Lemma 4.

and proceeding as in the proof of lemma 1, we can calculate a circle $C'(C',r')$ larger than C and tangent to P_0 at A. More analyticly, $C'\in\overline{CA'}$ and $r'=r+(\epsilon^*/2)$, where $\epsilon^*=\min\{d(C,x):x\in P_1\}-r>0$, A' antidiametrical to A, and P_1 the part of the polygon which lies in the subplane R_1 defined by the tangent ℓ (to C at A) and the center C. The initial assumption is contradicted by the fact that $r'>r$.

Theorem 2. The center of a circle $C(C,r)$, inscribed in a polygon P, belongs to the trajectory of the RC center, if and only if, C_0 has at least two common (tangential) points with P_0.

Proof

It is clear that, to every point of the trajectory, corresponds a RC being tangent to P_0 at, at least, two points (lemma 4). We prove now the reverse, that is, if an inscribed circle possesses two or more tangential points (in common with P_0), then its center lies on the trajectory.

Let A be one of the two tangential points (B the other -figure 14) and ℓ the straight line tangent to C at A. Then, point B lies in the interior of any circle centered on $\overleftarrow{AC}$, larger than C, and tangent to ℓ at A. But B belongs

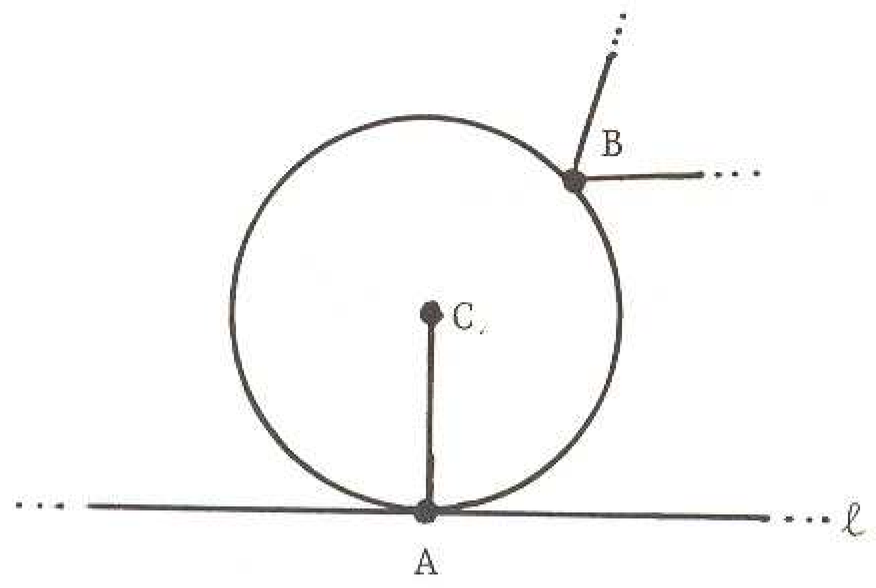

Figure 14. Theorem 2.

to P_0 and, hence, it cannot be interior point of an inscribed circle. Thus, C is the maximum circle, among those inscribed in the polygon and tangent to ℓ at A. Consequently, its center belongs to the trajectory of the RC center.

Theorem 3. (theorem of the tree-like trajectory). The trajectory of the RC center, in a polygon, consists of segments joined together in a tree-like manner, ie forming no loops.

Note: The trajectory of the RC center, in a convex polygon, consists solely of straight line segments, joined together in a tree-like manner.

Proof

We will prove that the trajectory has no loops. According to theorem 2, for any polygon (convex or not), the RC position, at a given moment of the rolling procedure, is such that there always exist two (or more) tangential points , on the perimeter of the polygon. In particular, if only two such points exist, then BTP is one of them and PTP the other.

Let bt and pt be the lines, tangent to the current RC, at BTP and PTP respectively (figure 15). Then, by definition, the current RC is the maximum circle, inscribed in the polygon and tangent to bt at BTP.

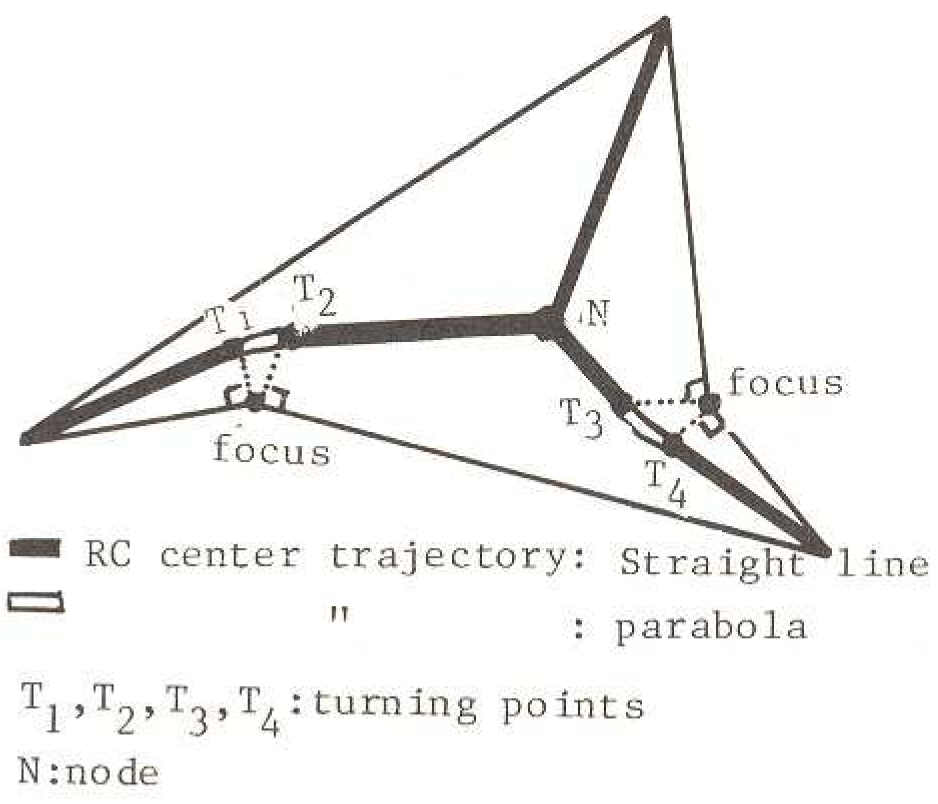

Figure 15. Theorem 3.

On the other hand (theorem 2 again), the very same RC is the maximum inscribed circle, among those tangent to pt at PTP. Considering, now, that RC is continuously "rolling" anti-clockwise, on the perimeter of the polygon, we easily realize that, either BTP is also moving anti-clockwise (if the base is a side), or RC is "turning" clockwise round a steady BTP (if the base is a non-convex vertex, in which case, base and BTP coincide). In either case, the current RC is "rolling" clockwise, in reference to the pilot, ie either PTP is moving clockwise (if the pilot is a side), or RC is turning anti-clockwise round a steady PTP (if the pilot is a non-convex vertex, in which case, pilot and PTP coincide). But, since every point of P_0 will eventually become a BTP, so will the current PTP. This means that, during the rolling procedure, the RC center will pass again from its current position, following the same track, but this time moving in the opposite direction: that s because the RC movement becomes anti-clockwise (in refernce to the current pilot) at some phase of the rolling procedure.

The above statement is true for every interior point of the trajectory which is not a knot. Whenever the RC center reaches a knot, it is forced, by the anti-clockwise RC movement, to follow the rightmost branch on its course (and also, to turn back, whenever it reaches a terminal point). So, if there was a loop, somewhere in the trajectory, then the RC center would be condemned to trace this loop only one way (either clockwise, or anti-clockwise, but not both). Thus, we would get a contradiction to the statement, that every segment of the trajectory is traced both ways.

In reference to the note, now, if the polygon is convex, then both the base and the pilot are always sides, and it has been shown that the parts of the trajectory, corresponding to such pairs, are all straight line segments.

4. The Algorithm

The algorithm requires an anti-clockwise enumeration of the set of vertices $V=\{V_1,V_2,\ldots,V_n\}$ and sides $S=\{s_1,s_2,\ldots,s_n\}$ of the polygon, considering, as first vertex (V_1) of V, the westmost one ($V_1=\{\text{vertex } V^*\in V: x(V^*)\leq x(V)$ for every $V\in V\}$) and, as first side (s_1) of S, the right-adjoint to V_1 side of the polygon.

Since sides as well as vertices may act as bases or pilots, in the formation of the trajectory of the RC center (a subject analyticly tackled in section 2.3), for simplification purposes we will not treat them differently in the algorithm. Instead, we introduce the notion of the polygon "element"(being side or vertex, according to the position of BTP or PTP, along the perimeter of the polygon), as the current assignment to the base or the pilot. So, we define the set of elements of the polygon $E=\{e_1(=V_1),e_2(=s_1),\ldots,e_{2n-1}(=V_n),$ $e_{2n}(=s_n)\}$ and the subsequent ordering $e_k<e_\lambda$ to mean that element e_λ lies "to the right" of e_k, along the perimeter of the polygon.

The algorithm takes advantage of the tree-like structure of the RC-center trajectory (see theorem 3), proceeding (actually jumping) from one turning point of this trajectory to the next, and checking each time if the new turning point is a node, that is, if it is a junction where two or more branches (and subsequent sub-trees) are emanating from. Note that this happens whenever the number of tangential points on the RC front (which is tangent to the perimeter of the polygon) are more than two, or equivalently, when the RC front collides with one or more barriers, not adjacent to the current base or pilot. Note, also, that V_1, is considered as the root node of the tree (that is, of the RC-center trajectory, whereas "father-of-node" and "son-of-node" are used to describe the relation between two successive nodes.

In respect, now, with the set $\{E_T\}$ of the tangential elements (tangent to the RC front), any two successive elements in E_T, define a "gate" leading to some part of the polygon, where a sub-tree of the RC-center trajectory is unfolded. Consider the example shown in figure 16. At the node N (of the tree -like trajectory), the RC front has three tangential elements, two of them being the current base and pilot, and the third (side AD) acting as barrier. The element of base and the barrier AD, together, define the first gate through which the search will be performed, whereas the barrier with the pilot define the second gate.

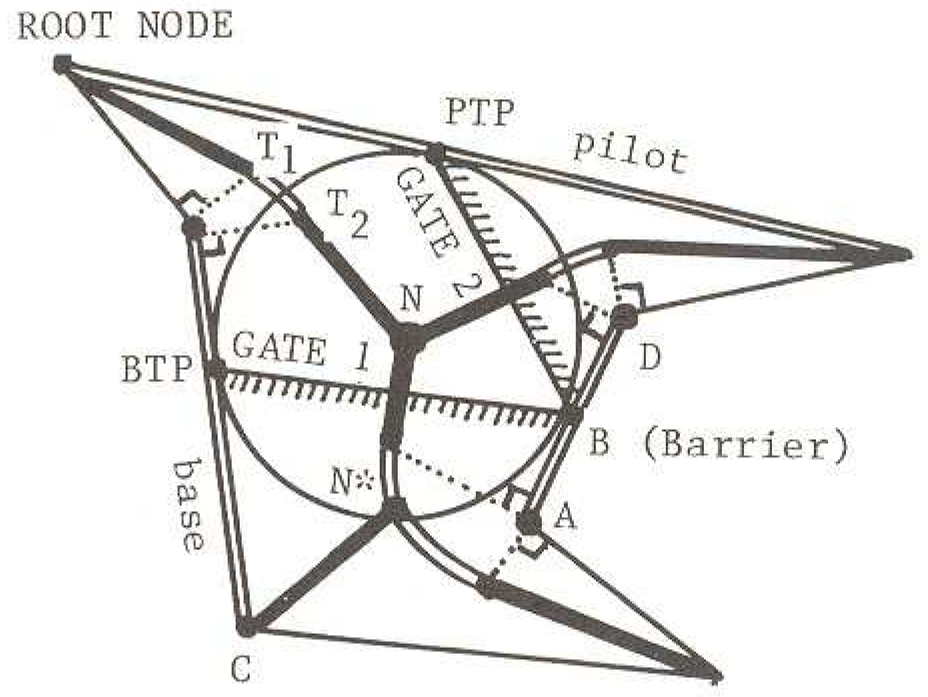

Figure 16. Theorem 3.

The scanning of sub-trees emanating from a particular junction (node n_c) will continue, until all the respective gates have explored. Then, as next node, the father-of-n_c will be considered (new n_c=father-of-n_c). The exploration, now, continues from on older position, with the remaining gates and subsequent sub-trees that have not yet been explored (corresponding to barriers which are still "active"). Furthermore, whenever the turning point T_c coincides with a convex vertex (reaches a terminal node of the tree -as is point C in figure 16), then the algorithm performs an inflection at this point, that is, sends T_c back to the position of the preceding node, because the path leading from that node to the terminal one (path N*C in figure 16) is considered as already scanned, and the associated gate as being non-active.

Finally, with respect to the updating process of the current best circle C_{max} it is reminded that there always exists a MIC having three or more tangential points (see theorem 1). This condition can be satisfied only at the nodes of the RC-center trajectory (because any turning point, which is not a node, corresponds to a phase of the rolling procedure, where exactly two tangential points exist). This suggests that the updating process should be applied, solely, at the nodes of the tree-like trajectory.

The algorithm NONVEX

Step. 1:"Initialization of the Algorithm"

S $=\{s_1, s_2, \ldots, s_n\}$: the set of sides of the polygon

V $=\{V_1, V_2, \ldots, V_n\}$:the set of vertices of the polygon

E $=\{e_1, e_2, \ldots, e_{2n}\}$:the set of elements of the polygon

base $=s_1$, pilot $=s_n$

C_{max}: ($C_{max}=V_1$, $r_{max}=0$) : the current best circle

$C_c=C_{max}$: the current circle

$T_c = V_1$: the current turning point

$n_c = V_1$: the current node (root node)

Step. 2:"Calculation of the next turning point"

2a: $E^* = \{e \in E :$ base $< e <$ pilot$\}$

set status-of-e ="active" $\forall\ e \in E^*$

2b: t =TRAJ (base, pilot) :TRAJ is the routine calculating the trajectory modulated by the current base and pilot (section 2.3)

2c: search for an "active" element $e \in E^*$

if no such element exists then go to step 2f

2d: t_c = TRAJ (base, e) :the candidate trajectory corresponding to the candidate barrier e (section 2.4)

$T_c^* = t_c \cap t$:the candidate turning point corresponding to candidate barrier e

C_ℓ : $(C_\ell = T_c^*,\ r_\ell = d(T_c^*, e))$, the limiting circle (see sections 2.4,2.5)

A_a = ACT (base, pilot, C_c front, C_ℓ front):ACT is the routine calculating the active area (section 2.5)

2e: search for an "active" element $e \in E^*$

if no such element exists then go to step 2f

else if $e \notin A_2$ then set status-of-e = "non-active"

and go back to step 2e

else go to step 2d

2f: update C_c and T_c

$C_c = C_\ell,\ T_c = T_c^*$

Step. 3:"Check for Terminal Nodes"

If T_c is terminal then set $T_c = n_c$ and go to step 5b $\left(\begin{array}{l}:n_c \text{ is the last node from} \\ \text{which } T_c \text{ is emanating}\end{array}\right)$

Step. 4:"Calculation of Tangential Elements of Current RC Front and for a Junction"

4a: calculate the set E_T of tangential elements of C_c front

4b: if $|E_T| > 2$ then T_c is a junction (and hence a node) so, update current best circle: $\left[\begin{array}{l}\text{if } r_c > r_{max} \\ \text{then set } C_{max} = C_c\end{array}\right]$ and go to step 5

else go to step 2

Step. 5:"Search of the Subtree Emanating from the Current Node"

5a: set son-of-$n_c = T_c$

$n_c \to T_c$

5b: order the tangential elements of E_T in an anti-clockwise manner, along the circumference of C_c, starting from the element of base and ending with the element of pilot

$E_T = \{e_{n_c\text{index}(1)} = \text{element of base}, e_{n_c\text{index}(2)}, \ldots$

$\ldots, e_{n_c\text{index}(\max n_c)} = \text{element of pilot}\}$

where $n_c$index(i) is the index of the i^{th} tangential element (with respect to node n_c) in the original set E.

set $i(n_c)=0$:the index of the element of base, leading to the next gate to be explored.

5c: $i(n_c)=i(n_c)+1$

if $i(n_c) = \max n_c$ then n_c=father-of-n_c $\left(\begin{array}{l}:\text{all gates emanating from} \\ n_c \text{ have been explored and}\end{array}\right)$

$T_c = n_c$

and go to step 5c $\left(\begin{array}{l}\text{.... thus we go back to the}\\ \text{father-of-}n_c\end{array}\right)$

else set base = $e_{n_c}\text{index}(i(n_c))$, pilot = $e_{n_c}\text{index}(i(n_c)+1)$

$\left(\begin{array}{l}\text{:explore the } i(n_c)\\ \text{gate of node } n_c\end{array}\right)$

If pilot = V_{n-1} or s_{n-1} then go to step 6

else go to step 2

Step. 6: "Terminate".

5. *Proposed Generalization of the Problem*

The analytic location of the RC center trajectory, in the proposed method, can offer the means for tackling the following generalized problem. Instead of having to locate the maximum circle(s) inscribed in a polygon (that is, to locate the obnoxious facility, as far as possible from the perimeter of the polygon), we ask for a location to be at a distance greater or equal to a given threshold security level s.

Problem NONVEX(s): find $c\in P$: $\min\{d(c,x):x\in P_0\}\geq s$

A simple method to solve this problem is to follow the course of the RC center trajectory, until we reach a point c satisfying the above inequality.

Consider now the general problem introduced in section 1. In this problem, call it G-NONVEX, we have a (closed) polygonal frontier-line surrounding a number of areas, characterized either "protected" or "restricted", which all have polygonal perimeters. By the term "protected", we mean an area, not only forbidden to build inside an obnoxious facility, but for which care should be taken, to locate the facility as far as possible from its perimeter. By "resticted", on the other hand, we mean simply an area forbidden to build inside it the obnoxious facility. The area outside the polygonal frontier-line is considered restricted and may also contain protected areas.

Given the above geometrical structure, we ask for the maximum circle centered inside a "free" area (an area, lying inside the frontier-line, which is neither protected nor restricted) and intersecting none of the protected areas.

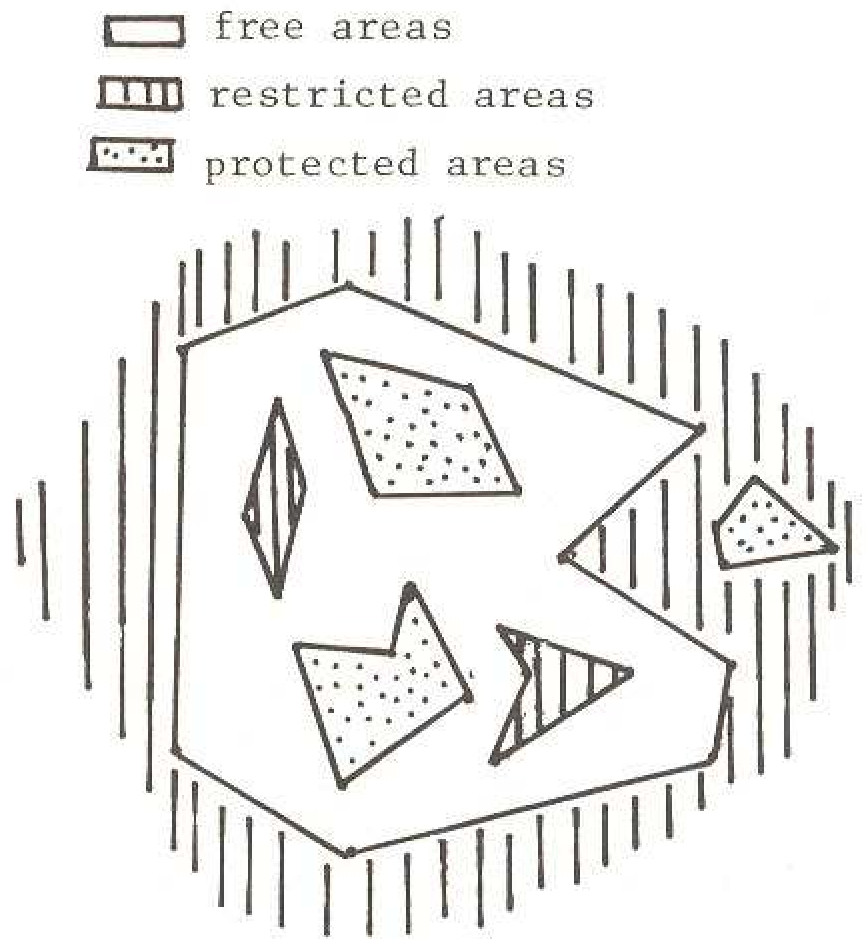

Figure 17. The General Problem.

Working currently on this problem, we have focused our effort on generalizing appropriately the basic concepts of the NONVEX algorithm and applying them on the G-NONVEX problem. The central idea is to use the notion of a local -maximum circle following a feasible course (ie intersecting no protected areas), while "rolling" around the perimeter of protected areas. There are indications that, in this case, the trajectory of the rolling circle defines a network, possibly including loops.

6. The order of the Algorithm Proposed.

Denote by m the number of non-convex vertices of the polygon. A turning point (which is not a node) appears on the trajectory of the RC center, whenever BTP (the base tangential point) or PTP (the pilot tangential point) come to, or leave from a non-convex vertex. Particularly, in the case BTP (or PTP) acts as a focus to the formation of a parabolic part of the trajectory, there are two turning points associated with that non-convex vertex: one at the beginning of the parbolic "rotation" and another at the end. Thus, the number of turning points (other than nodes), appearing on the trajectory of the RC center, cannot exceed 2m. On the other hand, it is well known that the number of nodes in a tree is less than the number of terminal points (here corresponding to convex vertices of the polygon). Hence, the number of nodes in the tree-like trajectory of the RC center cannot exceed n-m (the number of convex vertices), where n is the total number of vertices of the polygon. That brings the total number of turning points, including the nodes of the tree, to less than $2m+(n-m)\leq 2n$. Finally, each of these turning points can be associated with no more than n candidate turning points (corresponding to different candidate barriers) and, consequently, $2n^2$ is the total number of the operations required in the worst case. That is, the order of the algorithm proposed is $O(n^2)$.