STRESS VS STRENGTH PROBLEM

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ABSTRACT. — In this paper, some reliability models are presented for a stress vs. strength problem. A particular case, in which both stress and strength are normally distributed, has been discussed. The model has also been illustrated by solving a numerical example.

Introduction.

Scooman [4] developed some reliability models introducing a new variable 'z', such that $z = y - x$, where y and x stand for strength and stress, respectively. However, it is not always possible to find the combined effect of y and x i.e. z, specially when they (y and x) have different distributions.

Lipow [2] also developed a reliability model assuming the distribution of stress and strength to be normal and trucated normal, respectively. His approach seems to overestimate the reliability. With these in view, certain reliability models have been developed to enable to evaluate the reliability even in the case when stress and strength have different distributions.

Statement of the problem.

Consider two independent continuous random variables y $(0 \leqslant y \leqslant \infty)$ and x $(0 \leqslant x \leqslant \infty)$ representing strength and stress, respectively. It is assumed that the distributions of y and x are known and their probability density functions (p.d.fs.) are denoted by $f(y)$ and $\zeta(x)$, respectively. Since the failures of y occur only due to stress, therefore, it becomes necessary to measure both the variables in a common unit. Assume further that the two curves have been plotted on a common graph and P is the proportion of area common to both curves (fig. 1).

It will be worthwhile to note from the above graph that if there is any failure in y due to x, it must occur in the overlapped area between the two curves i.e., P. However, it does not mean that all the components whose strength falls in this region should be regarded as failures.

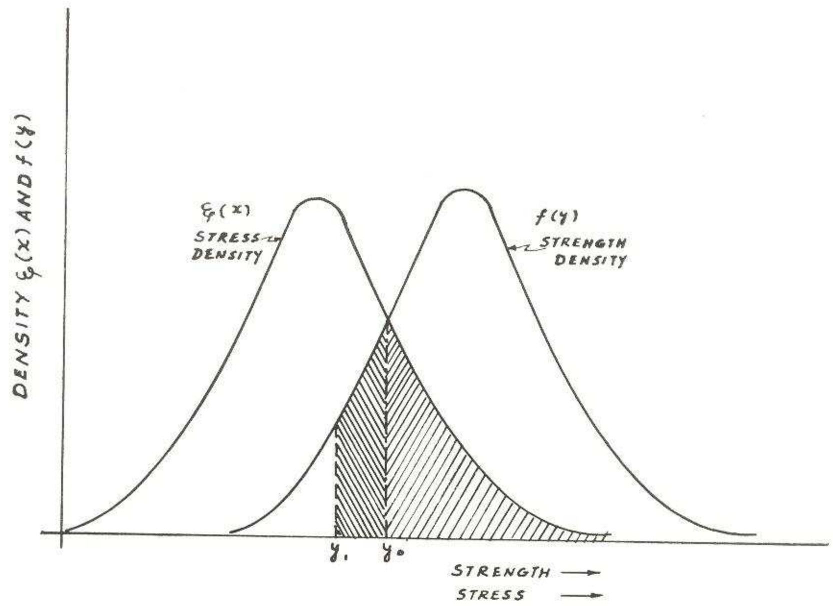

FIG. 1 STRESS . STRENGTH MODEL

If p' is the probability of failures of y, the reiability 'R' or the probability of success is given by

$$R = 1 - p', \qquad p' \leqslant P \tag{1}$$

Thus the value of R depends upon the value of p' and P and hence the present problem is to evaluate these parameters.

Development of the models.

Let N be the total number of identical components, out of which n lie in the overlapped area, denoted by y_i $(i = 1, 2, ..., n)$. Similarly, let x_i $(i = 1, 2, ..., n)$ be the values of stress in the area under consideration. Without loss of generality, we can assume $y_1 < y_2 < ... < y_n$ and $x_1 < x_2 < ... < x_n$, such that

$$x_i = y_i \text{ for all } i, \qquad (i = 1, 2, ..., n) \tag{2}$$

Now if we choose at random a value y_j out of n values, viz, $y_1, y_2, ..., y_n$, three possible situations occur, namely

$$\begin{aligned} &\text{(i)} \quad y_j < x_i \text{ for } i, \quad i = j+1, j+2, \ldots, n, \\ &\text{(ii)} \quad y_j > x_i \text{ for } i, \quad i = 1, 2, \ldots, j-1, \\ &\text{(iii)} \quad y_j = x_i \text{ for } j = i \end{aligned} \tag{3}$$

Obviously, y_j fails in case (i) and does not fail in case (ii), but in case (iii) it may or may not fail. In order to take a decision in case (iii), consider infinite number of such tie cases. Then one can definitely say that in approximately half of such cases, y will fail. Therefore, the probability that y_j fails when encountered by an equal amount of stress is approximately 0.5. Combining (i) and (iii) together we have $(n - j + .5)$ cases favourable to y_j for failure. The total number of cases, in which y_j can encounter stress x, is n, i.e. all the possible values of x lying in the overlapped area. Following Weatherburn [5] $p_f(y_j)$, the probablity that y_j fails when encountered by stress x is

$$p_f(y_j) = \frac{n - j + .5}{n} \qquad (j = 1, 2, \ldots, n) \tag{4}$$

and $p_s(y_j)$ the probability that it does not fail i.e. the probability of survival is given by

$$\begin{aligned} p_s(y_j) &= \frac{j - .5}{n} \\ &= 1 - p_f(y_j), \qquad (j = 1, 2, \ldots, n) \end{aligned} \tag{5}$$

Now, allowing y_j to vary over the overlapped area and attaching a value 1, to y_j if it fails, and 0 if it does not fail, we get

$$\mathrm{E}(n') = \mathrm{E} \sum_{j=1}^{n} (y_j) \tag{6}$$

where n' is the number of failures.

Using relations (4) and (5) in relation (6) we obtain

$$\begin{aligned} \mathrm{E}(n') &= \sum_{j=1}^{n} 1 \times \frac{n - j + .5}{n} + 0 \times \frac{j - .5}{n} \\ &= n/2 \\ &= 1/2 \times \text{overlapped area} \end{aligned} \tag{7}$$

Here p' and R are given by

$$\begin{aligned} p' &= n'/\mathrm{N} \\ &= n/2\,\mathrm{N} \end{aligned} \tag{8}$$

and

$$R = 1 - n/2N \tag{9}$$

Obviously in the probability sense the ratio n/N represents the area given by P under the curves. Therefore

$$p' = P/2 \tag{10}$$

and

$$R = 1 - P/2 \tag{11}$$

Evidently, when $\zeta(x)$ and $f(y)$ coincide, $n = N$, thus

$$p' = R = 1/2 \tag{12}$$

In order to evaluate p' and R, we now proceed to evaluate P the overlapped area in the following manner :

Let the curves $f(y)$ and $\zeta(x)$ intersect at a point y_0 (fig. 1), so that the overlapped area P is given by

$$P = \int_{y_0}^{\infty} \zeta(x)\, dx + \int_{-\infty}^{y_0} f(y)\, dy \tag{13}$$

Sometimes in practice the components are screened to reject weaker components. This increases the mean strength of the remaining components. Now, suppose that the distribution of the remaining components is given by $f'(y)$, such that

$$f'(y) = \frac{f(y)}{\int_{y_1}^{\infty} f(y)\, dy} \qquad y_1 \leqslant y \leqslant \infty \tag{14}$$

where y_1 is the point of truncation.

Using the relation (14) in relation (13) we have

$$P = \begin{cases} \int_{y_1}^{y_0} f'(y)\, dy + \int_{y_0}^{\infty} \zeta(x)\, dx & \text{if } y_0 > y_1 \qquad (15) \\ \int_{y_1}^{\infty} \zeta(x)\, dx & \text{if } y_0 < y_1 \qquad (16) \\ \int_{y_0}^{\infty} \zeta(x)\, dx & \text{if } y_0 = y_1 \qquad (17) \end{cases}$$

The models developed above viz. models given in relations (13) through (17) are most general whatever be the distribution of stress and strength. However, to increase the practical utility of these models a particular case is discussed below.

Particular case.

Let both stress and strength be normally distributed, such that

$$f(y) = \frac{1}{\sqrt{2\pi}\,\sigma_1} \exp\{-\frac{1}{2}(\frac{y-\mu_1}{\sigma_1})^2\} \quad 0 < y < \infty \tag{18}$$

and

$$\zeta(x) = \frac{1}{\sqrt{2\pi}\,\sigma_2} \exp\{-\frac{1}{2}(\frac{y-\mu_2}{\sigma_2})^2\} \quad 0 < x < \infty \tag{19}$$

where μ_i and σ_i ($i = 1, 2$) are mean and standard deviation, respectively. Since $f(y)$ and $\zeta(x)$ intersect at y_0, setting $x = y = y_0$ in relations (18) and (19) and equating them we have

$$K_1 y_0^2 + K_2 y_0 + K_3 = 0 \tag{20}$$

where

$$K_1 = \frac{\sigma_1^2 - \sigma_2^2}{\sigma_1^2 \sigma_2^2}$$

$$K_2 = 2\,\{\frac{\mu_1 \sigma_2^2 - \mu_2 \sigma_1^2}{\sigma_1^2 \sigma_2^2}\}$$

and

$$K_3 = \{\mu_2^2 \sigma_1^2 - \mu_1^2 \sigma_2^2\}/\sigma_1^2 \sigma_2^2 + 2 \log_e \{\sigma_2/\sigma_1\}$$

If $\sigma_1 \approx \sigma_2$, $K_1 \to 0$; so that y_0 is given by

$$y_0 = -K_3/K_2$$

$$= \frac{\mu_1 + \mu_2}{2} \tag{21}$$

If $K_1 \neq 0$, equation (20) gives two values of y_0. Since both stress and strength vary from 0 to ∞, only positive value of y_0 would be admissible.

Numerical example (*).

Consider the case of an empty solid propellant rocket motor which undergoes a proof pressure test by being pressurised usually with water to a given level of pressure, y_1. The cases which rupture as a result of this test are discarded, thereby increasing the average burst strength, of the remaining group which are then loaded with propellant and ultimately either test fired for motor lot acceptance or for operational use.

(*) Extracted from [2].

It is known that the peak rocket motor operating pressure x is normally distributed with mean $\mu_2 = 500$ psi and $\sigma_2 = 100$ psi. The proof pressure test on the case is to pressurise it to $y_1 = 600$ psi, when it is known that the mean case strength (pressure at which the case ruptures) is normally distributed with mean $\mu_1 = 700$ psi, and $\sigma_1 = 100$ psi.

Evidently, the distribution of strength is truncated at $y_1 = 600$ psi. On the assumption that the proof pressure test does not affect the strength of cases which are accepted for use, the point of intersection can be obtained by using the formula (21). Thus

$$y_0 = \frac{\mu_1 + \mu_2}{2} \qquad \text{since } \sigma_1 = \sigma_2$$

$$= 600 \text{ psi.}$$

Since $y_1 = y_0$, P is given by

$$P = \int_{y_1}^{\infty} \frac{1}{\sqrt{2\pi}\,\sigma_2} \exp\left\{-\frac{1}{2}\left(\frac{x-500}{100}\right)^2\right\} dx$$

$$= .1587$$

and

$$R = 1 - P/2 = .92065$$

The application of the models developed by Shooman [4] becomes difficult in this situation whereas according to Lipow [2] the value of R is .9683 which overestimates the reliability by 5.2 per cent as evaluated with the help of the model developed in this paper.

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