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A HOTELLING MODEL WITH DIFFERENTIAL WEIGHTS AND MOVING COSTS

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ABSTRACT

This paper examines a competitive spatial model on a linear market. The two competing facilities are assumed to have different weights. Based on a gravitational-type attraction function, the market areas of the two facilities are determined. Assuming individual optimization and allowing sequential relocations, it is shown that no locational Nash equilibrium exists. Then fixed and variable relocation costs are introduced. Necessary and sufficient conditions for both types of relocation costs are derived which guarantee that a stable equilibrium is reached.

INTRODUCTION

Ever since Hotelling (1929) introduced his spatial model, interest in competitive location models has flourished. The original Hotelling model consists of two equally attractive competitors and a linear market, i.e. a line segment on which both facilities are located. The customers are assumed to be uniformly distributed along the market and the demand is completely inelastic with respect to the price. The profit functions for both competitors are set up and the prices at optimum are derived. These functions indicate that both facilities best locate as close as possible to each other. Hotelling applied this result to various situations not all involving the location of physical facilities such as grocery stores but also locations of political candidates on a left-right scale, the location of products on a quality scale, etc. The result which states that under competition, facilities (whatever they may represent) locate close to each other at optimum, a situation termed "principle of minimal differentiation" by Boulding (1966).

The validity of this principle has frequently been challenged. Chamberlin (1933) observed that on a circular rather than linear market, clustering of the competitors no longer occurs. Lerner and Singer (1937) describe equilibrium location patterns for up to eight facilities given a fixed price. Their analysis was extended by Eaton and Lipsey (1975) who not only showed that the three-facility case is the only one which does not converge to an equilibrium but they also derive necessary and sufficient conditions for the existence of equilibrium of n facilities on a linear market. Carruthers (1981) generalized Eaton and Lipsey's results allowing different prices to be charged by the facilities. In his relocation and repricing procedure, price undercutting is prohibited. Generalized necessary and sufficient conditions for equilibrium are presented. Prohibiting price undercutting is necessary as d'Aspremont, Gabszewicz and Thisse (1979) have shown that otherwise the only stable solution is one where both facilities charge zero prices. Even more interesting, the authors show that given a quadratic (rather than linear) cost function, maximal and not minimal differentiation follows.

A number of other studies also suggests that the "principle of minimal differentiation" is by no means as general as originally assumed. Anderson (1987) uses Stackelberg's leader-follower concept and arrives at an equilibrium with one facility at the center and the other close to one of the ends of the market. Osborne and Pitchik (1987) find a pure equilibrium of the first-stage location game which locates the facilities close to the quartiles of the market, i.e. the social optima. Artle and Carruthers's (1988) land owner model finds optimal asymmetric locations. De Palma, Ginsburgh and Thisse (1987) as well as De Palma et al. (1985) introduce heterogeneity in customers' tastes. Their respective probabilistic models show that for any degree of heterogeneity, two facilities will not cluster at the center of the market.

The purpose of this paper is to show that another Hotelling-type model whose assumptions are very similar to the original one, does not only fail to show agglomeration at the center, but it does not result in an equilibrium at all. This amplifies the suspicion of the sensitivity of Hotelling's model to minor changes in the model. Subsequently, fixed and variable relocation costs are introduced and bounds for these costs are derived, so that a stable solution will eventually evolve.

THE MODEL AND ITS PROPERTIES

Suppose that the market is a line segment from 0 to 1 along which the demand is uniform. Let A and B be two facilities with weights w_A and w_B, respectively. The interpretation of weights depends on the particular model. If the facilities are shopping centers, they may express floor space, product variety, and relative price advantage. In a product (re-) design model, the weights could symbolize the non-quantifyable appeal of a product. Without loss of generality assume that $w_A > w_B$. Distances between any two points i and j are denoted by d_{ij}; for example d_{AB} is the distance between the two facilities, and d_{B1} is the distance of facility B from the right end of the market. Distances may be measured in either direction, i.e. $d_{ij} = d_{ji}$. A customer at point i is attracted to the two facilities according to the attraction functions $\phi^r_{Ai} = w_A/d^r_{Ai}$ and $\phi^r_{Bi} = w_B/d^r_{Bi}$ and he will patronize the facility he is more attracted to. If some customer-facility distance is zero, say $d_{Bi} = 0$, the customer will patronize facility B. If both facilities are located at the same point, all customers on the market patronize the larger facility. The attraction function used here is of the gravitational type.

A number of geographical studies use similar attraction functions but assume that customers patronize facilities in proportion to the magnitude of their attractions. As a result, a facility may used by customers anywhere on the market. In contrast, our use of the attraction function subdivides the given market into individual segments or market areas associated with the facilities. A customer located in the market area of, say, facility A will then exclusively patronize that facility. Market area models of this type have been used by Reilly (1929), Huff (1964), and Boots (1980). Note that for large values of r, the level of attraction drops off dramatically if one moves away from a facility. This indicates that distance is an important criterion for customers, i.e. the transportation cost are high.

Assume now that initially facility B is located to the left of the center of the market, i.e. $d_{0B} \leq \frac{1}{2}$, and let A be located to the right of B. The market areas of the two facilities can then be determined as follows. The smaller of the two facilities, B, has a market area which extends to the left so some point x and to the right to some point y. At x and y, customers are equally attracted to A and B. This situation is displayed in Figure 1.

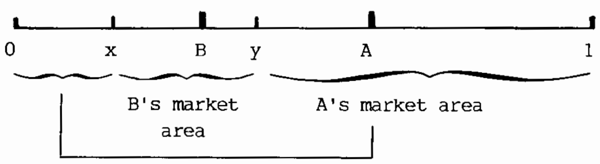

Figure 1

At y, we have $\phi^r_{Ay} = w_A/d^r_{Ay} = w_B/d^r_{By} = \phi^r_{By}$. With $d_{Ay} = d_{AB} - d_{By}$ we obtain

$$d_{By} = \frac{\sqrt[r]{w_B}}{\sqrt[r]{w_A} + \sqrt[r]{w_B}} \; d_{AB}. \tag{1}$$

Similarily, at x we have $\phi^r_{Ax} = w_A/d^r_{Ax} = w_B/d^r_{Bx} = \phi^r_{Bx}$. With

$d_{Ax} = d_{AB} + d_{Bx}$, we obtain $d_{Bx} = \sqrt[r]{w_B}\, d_{AB}/(\sqrt[r]{w_A} - \sqrt[r]{w_B})$. This relation is only valid as long as $d_{Bx} \le d_{0B}$. Hence

$$d_{Bx} = \min \left\{ d_{0B};\ \frac{\sqrt[r]{w_B}}{\sqrt[r]{w_A} - \sqrt[r]{w_B}}\, d_{AB} \right\} \tag{2}$$

Then the market (or Voronoi) area of facility B is

$$V(B) = d_{By} + d_{Bx} = \begin{cases} \dfrac{2\sqrt[r]{w_A w_B}}{\sqrt[r]{w_A} - \sqrt[r]{w_B}}\, d_{AB}, \text{ if } d_{AB} \le \dfrac{\sqrt[r]{w_A} - \sqrt[r]{w_B}}{\sqrt[r]{w_A}}\, d_{OA} \\[2ex] d_{OA} - \dfrac{\sqrt[r]{w_A}}{\sqrt[r]{w_A} + \sqrt[r]{w_B}}\, d_{AB} \quad \text{otherwise} \end{cases} \tag{3}$$

Note that is relation (3), facility B's only decision variable is d_{AB}, its location is relation to it competitor. In the first case, the coefficient of d_{AB} is positive so that B will choose the largest possible value of d_{AB}, i.e. $\dfrac{\sqrt[r]{w_A} - \sqrt[r]{w_B}}{\sqrt[r]{w_A}}\, d_{OA}$. In the second case, the coefficient of d_{AB} is negative hence B will keep d_{AB} as small as possible, leading again to

$$d_{AB} = \frac{\sqrt[r]{w_A} - \sqrt[r]{w_B}}{\sqrt[r]{w_A}}\, d_{OA}.$$ This leads to

Rule 1: B will locate in the half segment of the market not occupied by A in such a way that one boundary of V(B) coincides with an end of the market.

The movements of facility A are even easier to describe. Since the sum of the market areas of A and B is constant (=1), A tries to maximize V(A) = 1 - V(B) = 1 - (d_{Bx} + d_{By}) or, equivalently, minimize d_{Bx} + d_{By}. In the first case of relation (3), this is done by setting d_{AB} = 0. The second case can be rewritten as

$$V(B) = d_{OB} + \frac{\sqrt[r]{w_B}}{\sqrt[r]{w_A} - \sqrt[r]{w_B}} \, d_{AB}$$

and this is again minimized by setting d_{AB} = 0. This implies

Rule 2: Facility A will always locate as close as possible to B leaving B with virtually no market area.

Suppose now that the facilities relocate sequentially in some fixed order. If all facilities have relocated once, one "optimization round" is over. An example of the sequential relocation process is given in Figure 2 where w_A = 2, w_B = 1, the intial locations of A and B are 1/8 and 3/4, respectively, and A is the first facility to relocate.

The respective market shares for facilities A and B at the end of the iterations are as follows. Iteration 0: (0.6667, 0.3333), Iteration 1: 0.4167, 0.5833), Iteration 2: (0.625, 0.375), Iteration 3: (0.5208, 0.4792), Iteration 4: (0.5729, 0.4271).

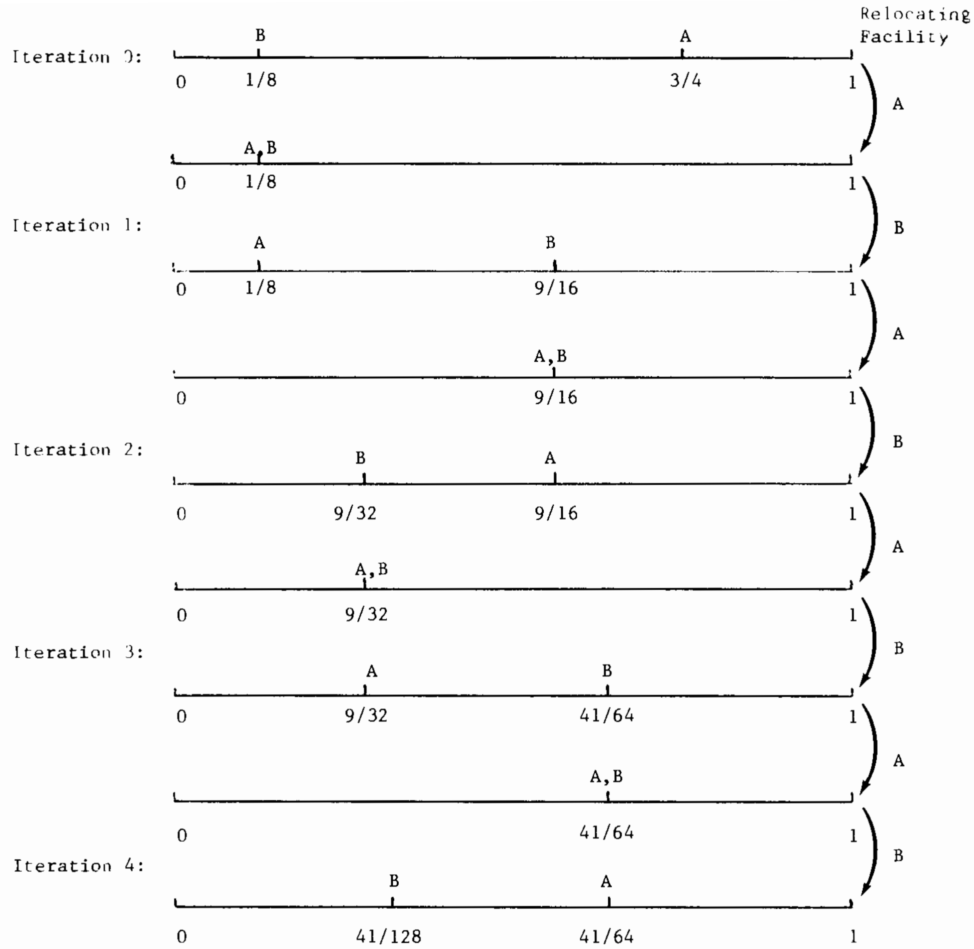
Relocating Facility
Iteration 0:
B
A
0
1/8
3/4
1
A
A,B
0
1/8
1
Iteration 1:
B
A
B
0
1/8
9/16
1
A
A,B
0
9/16
1
Iteration 2:
B
B
A
0
9/32
9/16
1
A
A,B
0
9/32
1
Iteration 3:
B
A
B
0
9/32
41/64
1
A
A,B
0
41/64
1
Iteration 4:
B
B
A
0
41/128
41/64
1

Figure 2

In order to find general expressions for the movements of A and B, define $d_{ij}^{(k)}$ as the distance between points i and j at the end of the k-th iteration. Then rule 1 implies

$$d_{0B}^{(k)} = \frac{\sqrt[r]{w_B}}{\sqrt[r]{w_A}} d_{0A}^{(k)} \text{ for } d_{0A}^{(k)} \geq 1/2 \tag{4}$$

and

$$d_{0B}^{(k)} = \frac{\sqrt[r]{w_A} - \sqrt[r]{w_B}}{\sqrt[r]{w_A}} + \frac{\sqrt[r]{w_B}}{\sqrt[r]{w_A}} d_{0A}^{(k)} \quad \text{for} \quad d_{0A}^{(k)} < 1/2 \tag{5}$$

On the other hand, rule two implies that

$$d_{0A}^{(k)} = d_{0B}^{(k-1)} \tag{6}$$

To further facilitate the discussion, define $d_{Be}^{(k)}$ as the distance between facility B and the closer end of the market at the end of iteration k, i.e.

$$d_{Be}^{(k)} = \min \left\{ d_{0B}^{(k)} \; ; \; d_{B1}^{(k)} \right\} \tag{7}$$

Then relations (4) and (5) can be written as

$$d_{Be}^{(k)} = \frac{\sqrt[r]{w_B}}{\sqrt[r]{w_A}} (1 - d_{Ae}^{(k)}) \tag{8}$$

and relation (6) is written as

$$d_{Ae}^{(k)} = d_{Be}^{(k-1)} \tag{9}$$

Inserting (9) in (8) yields the difference equation

$$d_{Be}^{(k)} = \frac{\sqrt[r]{w_B}}{\sqrt[r]{w_A}} (1 - d_{Be}^{(k-1)}) \tag{10}$$

Letting $k \longrightarrow \infty$, we obtain

$$d_{Be}^{(\infty)} = \frac{\sqrt[r]{w_B}}{\sqrt[r]{w_A} + \sqrt[r]{w_B}} \tag{11}$$

In other words, in the long run we will expect B to locate at a distance from the end of the market which is a function of its proportion of the total weight. Clearly, A will locate "on top of B" in its next move and B will then relocate at the same distance from the other end of the market. Thus a locational equilibrium will not be achieved, even though the facilities return to the same two locations step after step, suggesting the existing of what may be called a "distance equilibrium". Also note that for $w_B \simeq w_A$, relation (11) yields $d_{Be}^{(\infty)} \simeq \frac{1}{2}$ which happens to be the optimal location in the unweighted Hotelling model with a common price. On the other hand, for $w_A >> w_B$, the distance equilibrium locations are very close to the ends of the market. Most importantly however, even if the weights are just slightly different, no equilibrium exists. This further challenges the stability of the principle of minimal differentiation.

Another observation can be made with regard to relation (11). For $r \longrightarrow \infty$, $d_{Be}^{(\infty)}$ approaches $\frac{1}{2}$, meaning that facility B achieves its optimal location, a local monopoly, very close to facility A. This is due to the very high transportation costs. On the other hand, $r \rightarrow 0$ means cheap transportation and the smaller facility must move far away from the bigger one in order to remain attractive to at least some customers. It should be mentioned that planning with foresight à la Prescott and Visscher (1977), where each facility locates exactly once, does always lead to the first entry paradox; see Ghosh and Buchanan (1988). In other words, the facility that locates first is always at a disadvantage. For more details of this paradox on a tree, see Eiselt and Laporte (1991). It is also worth mentioning that two unequally weighted facilities on a circular market exhibit precisely the same behavior as those on a linear market. A recent result for equally weighted facilities in $\mathbb{R}^2$ was reported by Okabe and Suzuki (1987).

THE MODEL WITH MOVING COSTS

Having established the nonexistence of a Cournot - Nash equilibrium in our model we now introduce moving or relocation costs. Clearly, if the costs of relocating a facility outweigh the gain achieved by the relocation, the facility will simply not move. If this is true for all given facilities, an equilibrium has been reached. Assume now that the purchases of all customers on the market amount to one in each iteration. Then the market area of a facility equals its revenue and, considering relocation costs as the only cost type, moving costs which are at some point during the relocation process larger than the gain of the market area of any of the facilities stop any further movement. In this section we derive relocation costs which are necessary and sufficient to reach an equilibrium. Two types of costs are discussed, fixed and variable relocation costs. Fixed relocation costs are incurred whenever a facility moves, they are independent of the locations the facility is moved from and to. On the other hand, variable relocation costs are assumed to increase proportional to the distance a facility moves.

First consider the case of fixed relocation costs. Figure 3 depicts the locations and market areas of two facilities A and B in iterations $(k-1)$ and k, $k > 1$.

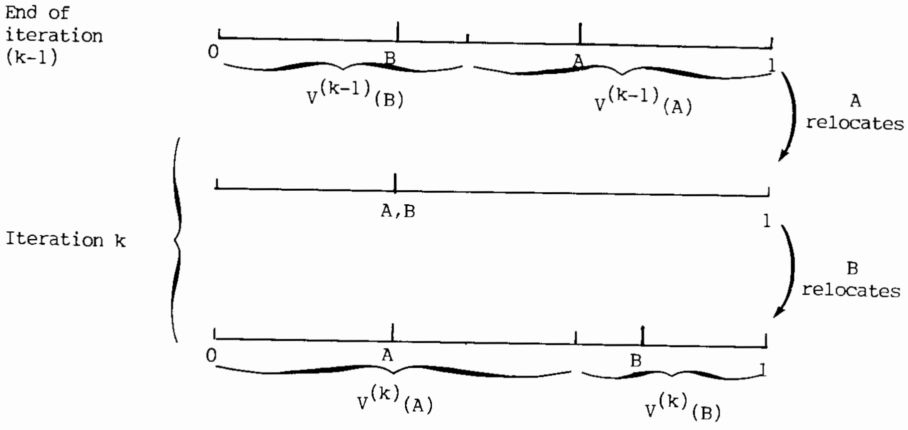

Figure 3

Figure 3 shows that facility A's gain getting into iteration k is $1-V^{(k-1)}(A) = V^{(k-1)}(B)$. Facility B's subsequent gain in iteration k is then $V^{(k)}(B)$. A's gain getting into iteration (k+1) will then again be $V^{(k)}(B)$ and so forth. Thus the sequence of the gains of facilities is

A: $V^{(0)}(B)$, B: $V^{(1)}(B)$, A: $V^{(1)}(B)$, B: $V^{(2)}(B)$, A: $V^{(2)}(B)$, ...

Note that except the very first move by facility A, the gains of the two facilities in two sequential moves are identical. Thus, if fixed moving costs C_f were introduced, so that $C_f \geq V^{(k)}(B)$ for some $k \geq 1$, neither facility has any reason to move and thus an equilibrium is reached. It is important to observe that two successive moves, one by A and one by B have to be blocked in order to stop all further relocations. This can be summarized in the following

Lemma 1: Fixed relocation costs $C_f \geq V^{(k)}(B)$, $k \geq 1$ are sufficient to reach an equilibrium.

For simplicity of the exposition we will assume that at least one of the facilities must relocate once. The reason for this assumption is technical as it eliminates the need to consider various cases of the arbitrary initial locations. Note that this implies that $d_{Bx} = d_{0B}$, so that only the first relation in (3) applies. Using the fact that $d_{AB}^{(k)} = 1 - d_{Ae}^{(k)} - d_{Be}^{(k)}$ as well as relation (9) we obtain

$$V^{(k)}(B) = \frac{2\sqrt[r]{w_B}}{\sqrt[r]{w_A} + \sqrt[r]{w_B}} (1 - d_{Be}^{(k-1)}) \tag{12}$$

In the following we will derive some results concerning the distance $d_{Be}^{(k)}$ which will enable us, by virtue of lemma and relation (12), to derive bounds for the relocation costs. For that purpose suppose that

$$d_{Be}^{(k-1)} \le d_{Be}^{(\infty)} \tag{13}$$

i.e. in iteration (k-1), facility B is located closer to the end of the market than at the distance equilibrium. Then

$$\begin{aligned} d_{Be}^{(k)} &= \frac{\sqrt[r]{w_B}}{\sqrt[r]{w_A}} - \frac{\sqrt[r]{w_B}}{\sqrt[r]{w_A}} d_{Be}^{(k-1)} && \text{due to (10)} \\ &\ge \frac{\sqrt[r]{w_B}}{\sqrt[r]{w_A}} - \frac{\sqrt[r]{w_B}}{\sqrt[r]{w_A}} \frac{\sqrt[r]{w_B}}{\sqrt[r]{w_A} + \sqrt[r]{w_B}} && \text{due to (11) and (13)} \\ &= d_{Be}^{(\infty)} \end{aligned}$$

or simply $d_{Be}^{(k)} \ge d_{Be}^{(\infty)}$. In other words, if facility B is farther away from to the end of the market in iteration (k-1) compared with the distance equilibrium, then it will be farther away from to an end of the market in iteration k. This implies

Lemma 2: $d_{Be}^{(k-1)} \leq d_{Be}^{(\infty)}$ implies that $d_{Be}^{(k)} \geq d_{Be}^{(\infty)}$ and similarly

$d_{Be}^{(k-1)} \geq d_{Be}^{(\infty)}$ implies that $d_{Be}^{(k)} \leq d_{Be}^{(\infty)}$.

But more can be said about the process with which the facilities approach the distance equilibrium. The ratio of the absolute deviation in the k-th iteration of the location from the distance equilibrium in relation to the absolute deviation of the location in the (k-1)st iteration is

$$\frac{\left| d_{Be}^{(k)} - d_{Be}^{(\infty)} \right|}{\left| d_{Be}^{(k-1)} - d_{Be}^{(\infty)} \right|} = \frac{\sqrt[r]{w_B}}{\sqrt[r]{w_B}} \qquad (14)$$

As an example, for $w_B/w_A = ½$ and $r = 1$, the deviation of the locations from the long-term distance equilibrium are cut in half in each iteration. Notice that the larger the difference between the two weights, the faster the convergence of the process. For $w_B \simeq w_A$, a case very similar to the original unweighted Hotelling case, the convergence will be very slow.

An example of the convergence process is given in Figure 4, where B^k denotes the location of facility B at the end of iteration k with B^∞ being the distance equilibrium.

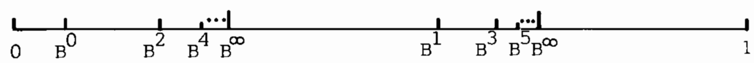

Figure 4

The above discussion implies that

$$d_{Be}^{(0)} \le d_{Be}^{(2)} \le d_{Be}^{(4)} \le \ldots \le d_{Be}^{(\infty)} \le \ldots \le d_{Be}^{(5)} \le d_{Be}^{(3)} \le d_{Be}^{(1)} \tag{15}$$

which, by virtue of relation (12), leads immediately to

$$V^{(1)}(B) \ge V^{(3)}(B) \ge V^{(5)}(B) \ge \ldots \ge V^{(\infty)}(B) \ge \ldots \ge V^{(4)}(B) \ge V^{(2)}(B) \tag{16}$$

Relation (16), coupled with lemma 1, leads to

Theorem 1: Fixed relocation costs of

$$C_f = V^{(2)}(B)$$

are necessary and sufficient to achieve a locational Nash equilibrium for two facilities with different weights given that at least one relocation is required.

Consider now variable relocation costs, in particular relocation costs, costs which are proportional to the distance moved in one step. Again, the gain of facility A when starting iteration (k+1), is $V^{(k)}(B)$, which is also B's gain in iteration k. The distance moved by B in iteration k is $(1 - d_{Be}^{(k-1)} - d_{Be}^{(k)})$ as can be visualized in Figure 3. Note that facility A moves the very same distance when getting into iteration (k+1). Thus with relocation costs proportional to the distance moved, we obtain for both facilities gains of $V^{(k)}(B)$ and costs of $C_v(1-d_{Be}^{(k-1)} - d_{Be}^{(k)})$. Then we can state

Lemma 3: For given variable moving costs C_v, an equilibrium will be reached if there exists some k>1, so that $V^{(k)}(B) - C_v\,(1-d_{Be}^{(k-1)} - d_{Be}^{(k)}) \le 0$.

Using relations (10) and (12), the sufficient condition in lemma 3 can be rewritten as

$$C_v \geq \frac{2\sqrt[r]{w_A w_B}}{\sqrt[r]{w_A^2} - \sqrt[r]{w_B^2}} \tag{17}$$

Note that the terms $d_{Be}^{(k-1)}$ have cancelled out and thus the bound on the variable relocation costs is independent of the location of B. Actually, all (gain)/(distance) ratios equal the right-hand side of relation (17). Thus we can state

Theorem 2: Variable moving costs of

$$C_v = \frac{2\sqrt[r]{w_A}\sqrt[r]{w_B}}{\sqrt[r]{w_A^2} - \sqrt[r]{w_B^2}}$$

are necessary and sufficient to reach a locational (Nash) equilibrium for two facilities with different weights given that at least one relocation is required.

Any variable relocation costs equalling or exceeding those specified in theorem 2 will stifle all movements immediately after facilities A and B have moved exactly once each. Note that for increasing w_A/w_B ratios the bound on C_v decreases. This implies that for $w_A \approx w_B$, i.e. a situation close to the original Hotelling case, the variable moving costs have to be very high if an equilibrium is ever to be reached. This indicates again the potential instability of the Hotelling model for small changes of the weights.

CONCLUSION

In this paper a competitive spatial model is described. It is an extension of the famous Hotelling problem with two facilities on a linear market. In the model under consideration customers are attracted to the facilities not only on the basis of distance but also on the weight of the facilities. It is shown that for two competing facilities on a linear market no Nash equilibrium exists. It is then proved that the successive locations approach two fixed points which have the same distance from the end points of the market. Then fixed and proportional moving costs are introduced and necessary and sufficient conditions for these costs are derived, so that the locational pattern reaches an equilibrium.

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