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## BELGISCH TIJDSCHRIFT VOOR STATISTIEK, INFORMATIEK EN OPERATIONEEL ONDERZOEK

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# **TRANSPORTATION TECHNIQUE FOR QUADRATIC FRACTIONAL PROGRAMMING**

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**ABSTRACT.** — In this paper a transportation technique for a quadratic fractional programming subject to linear constraints has been provided. In another article it has been shown that this function is pseudo monotonic which gives local optimum as global optimum.

## **Introduction**

This paper deals with a special type of problem which occurs in big business concerns to fill a number of vacancy categories which demand different capabilities, experiences and trainings. The applicants having different capabilities, experiences and trainings will have the value depending upon the jobs in which they are to be employed. It is always the sincere intention of the concern to assign the applicant categories to vacancy categories in such a way that the value of the objective function with which the business concern is dealing with is a maximum. The objective function considered here is pseudo monotonic [2].

This paper is the outcome of the main results of the paper [2] in which the author has proved that maximum will occur at the vertex of the feasible solution set and local maximum is global maximum. The present paper has been divided into three sections. In section 1, mathematical model is given. Preliminaries are given in section 2. Section 3 deals with the optimal conditions.

## **Section 1**

*Mathematical Model.*

Maximize

$$f(x) = \frac{\left( \sum_{i=1}^m \sum_{j=1}^n c_{ij} x_{ij} + \alpha \right)^2}{\left( \sum_{i=1}^m \sum_{j=1}^n d_{ij} x_{ij} + \beta \right)^2} \quad (1)$$

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subject to

$$\sum_{i=1}^m x_{ij} = b_j \quad j = 1, 2, \dots, n \quad (1.2)$$

$$\sum_{j=1}^n x_{ij} = a_i \quad i = 1, 2, \dots, m \quad (1.3)$$

$$x_{ij} \geq 0 \quad i = 1, 2, \dots, m \quad j = 1, 2, \dots, n \quad (1.4)$$

where  $x_{ij}$  = set of structural variables; these variables represent competitive candidates or activities.

$c_{ij}$ ,  $d_{ij}$  = set of profit coefficients in the problem and are the coefficients of the structural variables in the objective function.

## Section 2

### Preliminaries.

(i) The consistency condition for the existence of the solution to the problem is

$$\sum_{i=1}^m a_i = \sum_{j=1}^n b_j$$

In case  $\sum_{i=1}^m a_i < \sum_{j=1}^n b_j$ , then a fictitious personnel category  $\sum_{j=1}^n b_j - \sum_{i=1}^m a_i$  men is added to the problem. When  $\sum_{j=1}^n b_j < \sum_{i=1}^m a_i$ , then a fictitious job category containing  $\sum_{i=1}^m a_i - \sum_{j=1}^n b_j$  jobs is used [3].

(ii) These are in all  $m + n$  equations in (1.2) and (1.3), out of which always only  $m + n - 1$  are non-redundant i.e. any basis will involve only  $m + n - 1$  variables [4].

(iii) The set of feasible solutions is regular and non-empty.

(iv) Initial basic feasible solution can be found by using one of the well-known methods: North-West Corner Method, Volga's Method and Inspection Method [6].

(v) Simplex Multipliers. As in [7], we determine the simplex multipliers  $p_i^1$ ,  $p_i^2$  ( $i = 1, 2, \dots, m$ ) and  $q_j^1$ ,  $q_j^2$  ( $j = 1, 2, \dots, n$ ) from equations

$$c_{ij} + p_i^1 + q_j^1 = 0 \quad (2.1)$$

$$d_{ij} + p_i^2 + q_j^2 = 0 \quad (2.2)$$

$i, j$  take suffixes of basic variables

Further let

$$c_{ij}^1 = c_{ij} + p_i^1 + q_j^1 \quad (2.3)$$

$$d_{ij}^1 = d_{ij} + p_i^2 + q_j^2 \quad (2.4)$$

It must be noticed that we are dealing with a system of  $m + n - 1$  equations out of  $m + n$  equations given in (1.2) and (1.3) as one equation is always redundant. The choice of the redundant equation is immaterial, we may set arbitrarily one of the  $p_i$  or one of the  $q_j$  equal to zero and solve for the remaining  $m + n - 1$  simplex multipliers. These simplex multipliers would be unique as the set of equations (1.2) and (1.3) are independent. We shall make use of these values of the simplex multipliers in (2.3) and (2.4) to determine  $c_{ij}^1$  and  $d_{ij}^1$  for the non-basic variables.

### Section 3

We shall determine here the next best basic feasible solution which improves the value of the objective function. Objective function would be written in terms of non-basic variables only. The function  $f(X)$  is

$$f(X) = \frac{(\sum_{i=1}^m \sum_{j=1}^n c_{ij} x_{ij} + \alpha)^2}{(\sum_{i=1}^m \sum_{j=1}^n d_{ij} x_{ij} + \beta)^2}$$

$$= \frac{[\sum_{i=1}^m \sum_{j=1}^n c_{ij} x_{ij} + \sum_{i=1}^m p_i^1 (\sum_{j=1}^n x_{ij} - a_i) + \sum_{j=1}^n q_j^1 (\sum_{i=1}^m x_{ij} - b_j) + \alpha]^2}{[\sum_{i=1}^m \sum_{j=1}^n d_{ij} x_{ij} + \sum_{i=1}^m p_i^2 (\sum_{j=1}^n x_{ij} - a_i) + \sum_{j=1}^n q_j^2 (\sum_{i=1}^m x_{ij} - b_j) + \beta]^2} \quad (3.1)$$

because of  $\sum_{j=1}^n x_{ij} = a_i$  and  $\sum_{i=1}^m x_{ij} = b_j$  whatever may be the values of  $p_i^1, p_i^2, q_j^1, q_j^2$ .

(3.1) can also be put in the form

$$f(X) = \frac{[\sum_{i=1}^m \sum_{j=1}^n (c_{ij} + p_i^1 + q_j^1) x_{ij} - \sum_{i=1}^m p_i^1 a_i - \sum_{j=1}^n q_j^1 b_j + \alpha]^2}{[\sum_{i=1}^m \sum_{j=1}^n (d_{ij} + p_i^2 + q_j^2) x_{ij} - \sum_{i=1}^m p_i^2 a_i - \sum_{j=1}^n q_j^2 b_j + \beta]^2} \quad (3.2)$$

$p_i^1, p_i^2, q_j^1, q_j^2$  are chosen such that

$$\left. \begin{aligned} c_{ij} + p_i^1 + q_j^1 &= 0 \\ d_{ij} + p_i^2 + q_j^2 &= 0 \end{aligned} \right\} \begin{aligned} &i, j \text{ take suffixes of the basic set} \\ &\text{of } m + n - 1 \text{ basic variables.} \end{aligned}$$

$$f(X) = \frac{\left[ \sum_{i,j \in S} c_{ij}^1 x_{ij} - \sum_{i=1}^m p_i^1 a_i - \sum_{j=1}^n q_j^1 b_j + \alpha \right]^2}{\left[ \sum_{i,j \in S} d_{ij}^1 x_{ij} - \sum_{i=1}^m p_i^2 a_i - \sum_{j=1}^n q_j^2 b_j + \beta \right]^2} \quad (3.3)$$

where  $S$  is a set of non-basic variables. Making use of the given basic feasible solution, the value of the objective function at that basic feasible solution becomes

$$\frac{\left[ -\sum_{i=1}^m p_i^1 a_i - \sum_{j=1}^n q_j^1 b_j + \alpha \right]^2}{\left[ -\sum_{i=1}^m p_i^2 a_i - \sum_{j=1}^n q_j^2 b_j + \beta \right]^2} = \frac{[T_1]^2}{[T_2]^2} \quad (3.4)$$

where

$$\begin{aligned} T_1 &= -\sum_{i=1}^m p_i^1 a_i - \sum_{j=1}^n q_j^1 b_j + \alpha \\ T_2 &= -\sum_{i=1}^m p_i^2 a_i - \sum_{j=1}^n q_j^2 b_j + \beta \end{aligned}$$

All the non-basic variables are zero at the initial basic feasible solution. Equation (3.3) can be rewritten as

$$f(X) = \frac{\left[ \sum_{i,j \in S} c_{ij}^1 x_{ij} + T_1 \right]^2}{\left[ \sum_{i,j \in S} d_{ij}^1 x_{ij} + T_2 \right]^2}$$

Now we choose  $x_{pq}$  variable to enter the basic set at a value  $W (> 0)$  and  $x_{lm}$  [7] variable departs from basic set i.e. becomes non-basic at zero level. The value of objective function becomes [1]  $\bar{f}(X)$  such that

$$\bar{f}(X) = \frac{[T_1 + W c_{pq}^1]^2}{[T_2 + W c_{pq}^2]^2}$$

In case  $f(X) > \bar{f}(X)$  (strictly) {when all basic feasible solutions are non degenerate} i.e.

$$\frac{[T_1 + W c_{pq}^1]^2}{[T_2 + W c_{pq}^2]^2} - \frac{[T_1]^2}{[T_2]^2} > 0 \quad (3.6)$$

we can say that the value of the objective function will improve.  
(3.6) becomes

$$[W^2 (c_{pq}^1)^2 + 2 T W c_{pq}] T_2^2 - [W^2 (d_{pq}^1)^2 + 2 T_2 W d_{pq}^1] T_1^2 > 0$$

(denominator of the objective function is positive always) or

$$[T_2 c_{pq}^1 - T_1 d_{pq}^1] [W \{T_2 c_{pq}^1 + T_1 d_{pq}^1\} + 2 T_1 T_2] > 0$$

$c_{pq}^1$  and  $d_{pq}^1$  refer to the original basic feasible solution. Let

$$\phi_{ij} = [T_2 c_{pq}^1 - T_1 d_{pq}^1] [W \{T_2 c_{pq}^1 + T_1 d_{pq}^1\} + 2 T_1 T_2]$$

and it can be calculated for non-basic variables if  $p_1^1, p_1^2, q_1^1, q_1^2$  are known for the same.

Here  $\phi_{pq} = \max \phi_{ij} (\phi_{ij} > 0)$  i.e. we choose the most positive  $\phi_{ij}$  to determine the variable  $x_{pq}$  to enter the basis. The variable  $x_{lm}$  which is to leave the basic set and the value of the basic variables in the new basis can be determined in the same way as in the case of transportation problem in linear programming.

In case all  $\phi_{ij} \leq 0$  we get the optimal solution of the given problem.

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## STRESS VS STRENGTH PROBLEM

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**ABSTRACT.** — In this paper, some reliability models are presented for a stress vs. strength problem. A particular case, in which both stress and strength are normally distributed, has been discussed. The model has also been illustrated by solving a numerical example.

### Introduction.

Scooman [4] developed some reliability models introducing a new variable 'z', such that  $z = y - x$ , where  $y$  and  $x$  stand for strength and stress, respectively. However, it is not always possible to find the combined effect of  $y$  and  $x$  i.e.  $z$ , specially when they ( $y$  and  $x$ ) have different distributions.

Lipow [2] also developed a reliability model assuming the distribution of stress and strength to be normal and truncated normal, respectively. His approach seems to overestimate the reliability. With these in view, certain reliability models have been developed to enable to evaluate the reliability even in the case when stress and strength have different distributions.

### Statement of the problem.

Consider two independent continuous random variables  $y$  ( $0 \leq y \leq \infty$ ) and  $x$  ( $0 \leq x \leq \infty$ ) representing strength and stress, respectively. It is assumed that the distributions of  $y$  and  $x$  are known and their probability density functions (p.d.fs.) are denoted by  $f(y)$  and  $\zeta(x)$ , respectively. Since the failures of  $y$  occur only due to stress, therefore, it becomes necessary to measure both the variables in a common unit. Assume further that the two curves have been plotted on a common graph and  $P$  is the proportion of area common to both curves (fig. 1).

It will be worthwhile to note from the above graph that if there is any failure in  $y$  due to  $x$ , it must occur in the overlapped area between the two curves i.e.,  $P$ . However, it does not mean that all the components whose strength falls in this region should be regarded as failures.

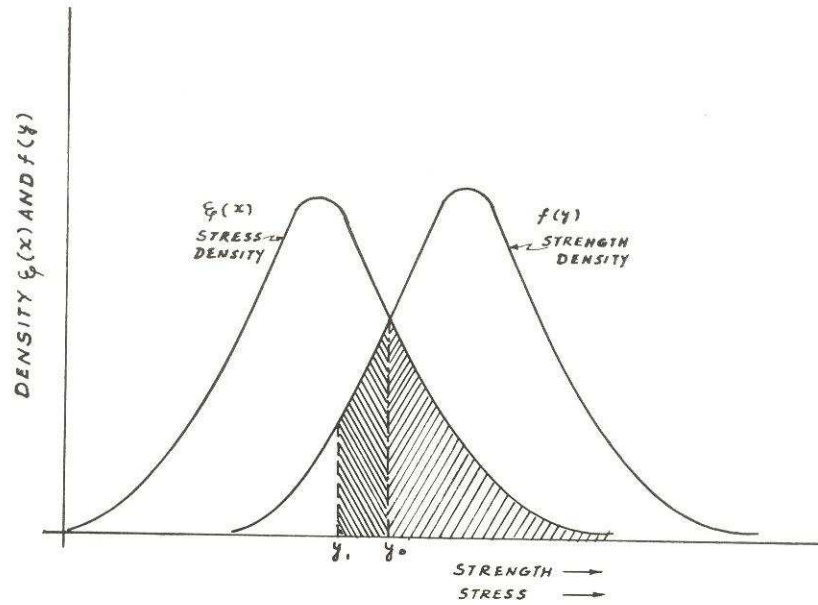


FIG.1 STRESS-STRENGTH MODEL

If  $p'$  is the probability of failures of  $y$ , the reliability 'R' or the probability of success is given by

$$R = 1 - p', \quad p' \leq P \quad (1)$$

Thus the value of R depends upon the value of  $p'$  and P and hence the present problem is to evaluate these parameters.

#### Development of the models.

Let N be the total number of identical components, out of which  $n$  lie in the overlapped area, denoted by  $y_i$  ( $i = 1, 2, \dots, n$ ). Similarly, let  $x_i$  ( $i = 1, 2, \dots, n$ ) be the values of stress in the area under consideration. Without loss of generality, we can assume  $y_1 < y_2 < \dots < y_n$  and  $x_1 < x_2 < \dots < x_n$ , such that

$$x_i = y_i \text{ for all } i, \quad (i = 1, 2, \dots, n) \quad (2)$$

Now if we choose at random a value  $y_j$  out of  $n$  values, viz,  $y_1, y_2, \dots, y_n$ , three possible situations occur, namely

- (i)  $y_j < x_i$  for  $i, i = j + 1, j + 2, \dots, n,$
  - (ii)  $y_j > x_i$  for  $i, i = 1, 2, \dots, j - 1,$
  - (iii)  $y_j = x_i$  for  $j = i$
- (3)

Obviously,  $y_j$  fails in case (i) and does not fail in case (ii), but in case (iii) it may or may not fail. In order to take a decision in case (iii), consider infinite number of such tie cases. Then one can definitely say that in approximately half of such cases,  $y$  will fail. Therefore, the probability that  $y_j$  fails when encountered by an equal amount of stress is approximately 0.5. Combining (i) and (iii) together we have  $(n - j + .5)$  cases favourable to  $y_j$  for failure. The total number of cases, in which  $y_j$  can encounter stress  $x$ , is  $n$ , i.e. all the possible values of  $x$  lying in the overlapped area. Following Weatherburn [5]  $p_f(y_j)$ , the probability that  $y_j$  fails when encountered by stress  $x$  is

$$p_f(y_j) = \frac{n - j + .5}{n} \quad (j = 1, 2, \dots, n) \quad (4)$$

and  $p_s(y_j)$  the probability that it does not fail i.e. the probability of survival is given by

$$\begin{aligned} p_s(y_j) &= \frac{j - .5}{n} \\ &= 1 - p_f(y_j), \quad (j = 1, 2, \dots, n) \end{aligned} \quad (5)$$

Now, allowing  $y_j$  to vary over the overlapped area and attaching a value 1, to  $y_j$  if it fails, and 0 if it does not fail, we get

$$E(n') = E \sum_{j=1}^n (y_j) \quad (6)$$

where  $n'$  is the number of failures.

Using relations (4) and (5) in relation (6) we obtain

$$\begin{aligned} E(n') &= \sum_{j=1}^n 1 \times \frac{n - j + .5}{n} + 0 \times \frac{j - .5}{n} \\ &= n/2 \\ &= 1/2 \times \text{overlapped area} \end{aligned} \quad (7)$$

Here  $p'$  and  $R$  are given by

$$\begin{aligned} p' &= n'/N \\ &= n/2N \end{aligned} \quad (8)$$

and

$$R = 1 - n/2N \quad (9)$$

Obviously in the probability sense the ratio  $n/N$  represents the area given by  $P$  under the curves. Therefore

$$p' = P/2 \quad (10)$$

and

$$R = 1 - P/2 \quad (11)$$

Evidently, when  $\xi(x)$  and  $f(y)$  coincide,  $n = N$ , thus

$$p' = R = 1/2 \quad (12)$$

In order to evaluate  $p'$  and  $R$ , we now proceed to evaluate  $P$  the overlapped area in the following manner :

Let the curves  $f(y)$  and  $\xi(x)$  intersect at a point  $y_0$  (fig. 1), so that the overlapped area  $P$  is given by

$$P = \int_{y_1}^{\infty} \xi(x) dx + \int_{-\infty}^{y_0} f(y) dy \quad (13)$$

Sometimes in practice the components are screened to reject weaker components. This increases the mean strength of the remaining components. Now, suppose that the distribution of the remaining components is given by  $f'(y)$ , such that

$$f'(y) = \frac{f(y)}{\int_{y_1}^{\infty} f(y) dy} \quad y_1 \leq y \leq \infty \quad (14)$$

where  $y_1$  is the point of truncation.

Using the relation (14) in relation (13) we have

$$\int_{y_1}^{y_0} f'(y) dy + \int_{y_0}^{\infty} \xi(x) dx \quad \text{if } y_0 > y_1 \quad (15)$$

$$P = \begin{cases} \int_{y_1}^{\infty} \xi(x) dx & \text{if } y_0 < y_1 \\ \int_{y_0}^{\infty} \xi(x) dx & \text{if } y_0 = y_1 \end{cases} \quad (16)$$

$$\int_{y_0}^{\infty} \xi(x) dx \quad \text{if } y_0 = y_1 \quad (17)$$

The models developed above viz. models given in relations (13) through (17) are most general whatever be the distribution of stress and strength. However, to increase the practical utility of these models a particular case is discussed below.

**Particular case.**

Let both stress and strength be normally distributed, such that

$$f(y) = \frac{1}{\sqrt{2\pi}\sigma_1} \exp \left\{ -\frac{1}{2} \left( \frac{y - \mu_1}{\sigma_1} \right)^2 \right\} \quad 0 < y < \infty \quad (18)$$

and

$$\xi(x) = \frac{1}{\sqrt{2\pi}\sigma_2} \exp \left\{ -\frac{1}{2} \left( \frac{x - \mu_2}{\sigma_2} \right)^2 \right\} \quad 0 < x < \infty \quad (19)$$

where  $\mu_i$  and  $\sigma_i$  ( $i = 1, 2$ ) are mean and standard deviation, respectively. Since  $f(y)$  and  $\xi(x)$  intersect at  $y_0$ , setting  $x = y = y_0$  in relations (18) and (19) and equating them we have

$$K_1 y_0^2 + K_2 y_0 + K_3 = 0 \quad (20)$$

where

$$K_1 = \frac{\sigma_1^2 - \sigma_2^2}{\sigma_1^2 \sigma_2^2}$$

$$K_2 = 2 \left\{ \frac{\mu_1 \sigma_2^2 - \mu_2 \sigma_1^2}{\sigma_1^2 \sigma_2^2} \right\}$$

and

$$K_3 = \{\mu_2^2 \sigma_1^2 - \mu_1^2 \sigma_2^2\} / \sigma_1^2 \sigma_2^2 + 2 \log_e \{\sigma_2 / \sigma_1\}$$

If  $\sigma_1 \approx \sigma_2$ ,  $K_1 \rightarrow 0$ ; so that  $y_0$  is given by

$$\begin{aligned} y_0 &= -K_3 / K_2 \\ &= \frac{\mu_1 + \mu_2}{2} \end{aligned} \quad (21)$$

If  $K_1 \neq 0$ , equation (20) gives two values of  $y_0$ . Since both stress and strength vary from 0 to  $\infty$ , only positive value of  $y_0$  would be admissible.

**Numerical example (\*).**

Consider the case of an empty solid propellant rocket motor which undergoes a proof pressure test by being pressurised usually with water to a given level of pressure,  $y_1$ . The cases which rupture as a result of this test are discarded, thereby increasing the average burst strength, of the remaining group which are then loaded with propellant and ultimately either test fired for motor lot acceptance or for operational use.

(\*) Extracted from [2].

It is known that the peak rocket motor operating pressure  $x$  is normally distributed with mean  $\mu_2 = 500$  psi and  $\sigma_2 = 100$  psi. The proof pressure test on the case is to pressurise it to  $y_1 = 600$  psi, when it is known that the mean case strength (pressure at which the case ruptures) is normally distributed with mean  $\mu_1 = 700$  psi, and  $\sigma_1 = 100$  psi.

Evidently, the distribution of strength is truncated at  $y_1 = 600$  psi. On the assumption that the proof pressure test does not affect the strength of cases which are accepted for use, the point of intersection can be obtained by using the formula (21). Thus

$$\begin{aligned} y_0 &= \frac{\mu_1 + \mu_2}{2} \quad \text{since } \sigma_1 = \sigma_2 \\ &= 600 \text{ psi.} \end{aligned}$$

Since  $y_1 = y_0$ ,  $P$  is given by

$$\begin{aligned} P &= \int_{y_1}^{\infty} \frac{1}{\sqrt{2\pi}\sigma_2} \exp \left\{ -\frac{1}{2} \left( \frac{x - 500}{100} \right)^2 \right\} dx \\ &= .1587 \end{aligned}$$

and

$$R = 1 - P/2 = .92065$$

The application of the models developed by Shooman [4] becomes difficult in this situation whereas according to Lipow [2] the value of  $R$  is .9683 which overestimates the reliability by 5.2 per cent as evaluated with the help of the model developed in this paper.

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# EXACT INVESTIGATION OF ALL EFFECTS FOR EXTENSIONS OF ONE-WAY ANOVA MODEL WITH FIXED EFFECTS

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**ABSTRACT.** — Consider the standard one-way analysis of variance model with fixed effects. The customary assumptions of zero mean, no correlation, and equal variance are made for the random «error» terms in this balanced model. The usual normality assumption is also made when tests or confidence regions are desired. Extensions are made of this one-way model by addition of further «error» terms of one or two kinds. The extended models apply to much more general situations than does the standard model. However, exact procedures are obtained for investigating all the effects that appear in the standard model, and for investigating subsets of these effects. The generality level of an extension depends on what effects are investigated. For several extensions, the customary results for the standard model remain applicable. Some procedures differing from the customary ones are used for the other extensions. An application of the standard model to rejection of outlying observations is described and shown to be usable for some extensions.

## Introduction.

The balanced fixed effects model for one-way analysis of variance is

$$y_{jk} = \mu + \alpha_j + e_{jk} \quad (1)$$

where  $j = 1, \dots, J$  and  $k = 1, \dots, K$ , with  $J, K \geq 2$ . Here,  $y_{jk}$  is an observed random variable,  $\mu$  is a parameter,  $\alpha_j$  is a parameter such that  $\alpha_1 + \dots + \alpha_J = 0$ , and  $e_{jk}$  is an unobserved random variable. The  $e_{jk}$  are assumed to be uncorrelated with zero expectation and the same positive variance  $\sigma^2$ . They are also assumed to be from a normal distribution when something other than a point estimate is desired.

Model (1) provides a basis for investigating  $\mu$ ,  $\sigma^2$ , and the  $\alpha_j$  by point estimates, significance tests, and confidence regions. Moreover, as shown next, it furnishes a basis for deciding on rejection of outlying observations. An attractive feature is that significance levels, confidence coefficients, unbiasedness of estimates, etc. are determined exactly.

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Now, consider the outline of a basis for rejection of outlying observations when model (1) holds. Suppose that  $y_{jk*}$  is selected to be investigated as a possible outlier (without information about the observed values). That is, the possibility that  $e_{jk*}$  does not have the same distribution as the other  $e_{jk}$  is investigated. First, without knowledge of the observation values, divide the observations with this value of  $j$  into sets of size two, and zero or one set of size three (with  $y_{jk*}$  not in a set of size three). Any set of size three is converted to a set of size two, by adding two of its observations and dividing this sum by  $\sqrt{2}$  (to yield one « observation »). The resulting « observation » is denoted by  $y_{jk'}$ , where  $k'$  is the smaller of the values for  $k$  in the two observations summed. For each set (now all of size two), a statistic of the type

$$y_{jk(1)} - y_{jk(2)}$$

is formed, where  $k(1) = k^*$  for the set containing  $y_{jk*}$ . The resulting statistics are uncorrelated with zero mean and the same variance. Using the normality assumption, the statistic containing  $y_{jk*}$  can be investigated by a procedure for examining whether a specified (chosen without information about the observation values) observation, supposedly in a sample from a normal population with zero mean, is an outlier.

This paper gives seven extensions of the standard model (1). An extension is made by adding more « error » terms, of one or two kinds, to the standard model for  $y_{jk}$ . An extension occurs such that  $\mu$ ,  $\sigma^2$ , and one or more of the  $\alpha_j$  can all be investigated by exact procedures. Some of the statistics for these investigations differ from those customarily used for model (1). Other extensions are developed for investigating  $\sigma^2$ , for investigating  $\mu$  and  $\sigma^2$ , and for investigating  $\sigma^2$  and one or more of the  $\alpha_j$ . Only the customary results need be used for some of these extensions but some different procedures occur for the others. These extensions are based on an approach given in [1].

The extended models are usable for much more general situations than is the standard model. However, some extended models are much more generally applicable than others. The level of generality for an extended model depends on what is investigated. For example, the extended model for investigating all the types of effects is less generally applicable than models for investigating subsets of the types of effects (when there is no restriction on the eligible procedures). Also, the generality level of an extended model depends on whether, or not, all the investigation procedures are to be those customarily used for model (1). The generality level decreases when the eligible procedures are limited to the customary ones.

The extensions of model (1) are stated and discussed in the next section. Investigation procedures for use with these various extensions are outlined in the final section.

### Extended models.

Seven extensions of model (1) are given. These depend on which effects are to be investigated and on whether the statistical procedures are restricted to those customarily used for model (1).

First, consider the extended model when  $\mu$ ,  $\sigma^2$ , and one or more of the  $\alpha_j$  are all to be investigated. Limitation to the results customarily used does not apply to this case. The model is

$$y_{jk} = \mu + \alpha_j + e_{jk} + e'_k \quad (2)$$

where  $\mu$ ,  $\alpha_j$ , and  $e_{jk}$  have the same properties as for model (1). The additional random variables  $e'_1, \dots, e'_K$  must sum to zero but otherwise can have an arbitrary joint distribution. Also,  $e'_1, \dots, e'_K$  can have any allowable dependences with the  $e_{jk}$  (the dependence can be different for each combination of an  $e'_k$  with an  $e_{jk}$ ). The values of the  $e'_k$  are contributions imposed on the observations by the experimental circumstances, with different  $e_{jk}$  possibly having different influence on the random values that occur for  $e'_1, \dots, e'_K$ .

Second, consider the case where the eligible procedures are not restricted and investigation of  $\sigma^2$  and one or more of the  $\alpha_j$  is to occur. The extended model for this case is

$$y_{jk} = \mu + \alpha_j + e_{jk} + e_k^* \quad (3)$$

where, as throughout the remaining material,  $\mu$ ,  $\alpha_j$ , and  $e_{jk}$  have the same properties as in model (1). The additional random variables  $e_1^*, \dots, e_K^*$  do not necessarily sum to zero and can have an arbitrary joint distribution. Also, they can have any permissible dependences with the  $e_{jk}$ . Model (3) is an extension of model (2).

Third, consider the case where the procedures are restricted to these customarily used for model (1) and investigation of  $\sigma^2$  and one or more of the  $\alpha_j$  is to occur. The extension for this case is

$$y_{jk} = \mu + \alpha_j + e_{jk} + e' \quad (4)$$

where  $e'$  can have an arbitrary distribution and any permissible dependences with the  $e_{jk}$  (the dependence can be different for each  $e_{jk}$ ). This is the least general of the extensions considered and, for  $K > 2$ , is much less general than model (2).

Fourth, consider the case where the eligible procedures are not limited and both  $\mu$  and  $\sigma^2$  are investigated. The extended model is

$$y_{jk} = \mu + \alpha_j + e_{jk} + e'_k + e''_j \quad (5)$$

where  $e'_1 + \dots + e'_K = 0$  and  $e''_1 + \dots + e''_J = 0$  but otherwise the additional random variables can have an arbitrary joint distribution. Also they can have any allowable dependences with the  $e_{jk}$ . Model (5), which extends models (2) and (4), has a high level of generality.

Fifth, consider the case where the procedures are restricted to the customary ones for model (1) and investigation of both  $\mu$  and  $\sigma^2$  occurs. The extension is

$$y_{jk} = \mu + \alpha_j + e_{jk} + e''_j \quad (6)$$

where  $e''_1 + \dots + e''_J = 0$  but otherwise these random variables can have an arbitrary joint distribution. Also,  $e''_1, \dots, e''_J$  can have any permissible dependences with the  $e_{jk}$ . Model (5) also is a substantial extension of model (6).

Only  $\sigma^2$  is investigated for the final two extensions. When the eligible procedures are not restricted, the extension is

$$y_{jk} = \mu + \alpha_j + e_{jk} + e_k^{**} + e_j^{**} \quad (7)$$

where the additional  $J + K$  random variables can have an arbitrary joint distribution. Also, they can have allowable dependences with the  $e_{jk}$ . This is the most general model considered and is an extension of model (5).

Finally, suppose that the procedures are limited to those customarily used and  $\sigma^2$  is investigated. The extended model is

$$y_{jk} = \mu + \alpha_j + e_{jk} + e_j^{**} \quad (8)$$

where  $e_1^{**}, \dots, e_J^{**}$  can have an arbitrary joint distribution, and any allowable dependences with the  $e_{jk}$ . Model (8) has substantially less generality than model (7) but is the most general extension given that is usable with the method for rejection of outliers (as is easily seen).

As noted in the discussions of the extensions, the generality of a model is strongly reduced when the eligible procedures are limited to those customarily used for model (1). Actually, the only statistic encountered that is not customarily used for model (1) is the statistic for investigating  $\sigma^2$ . This statistic also occurs in tests and confidence intervals for investigating the other effects.

### Basis for investigations.

The forms of the statistics used for an investigation identify the extended model that is appropriate for this investigation. That is, the extension should be as general as possible, subject to all statistics used being such that the additional random variables for the extension do not occur (cancel out or sum out).

This section states the statistics considered for possible use along with the effects they investigate, the extended model(s) for which the additional terms do not occur, and pertinent properties. For a given investigation, at least one statistic is introduced for each type of effect ( $\mu$ ,  $\sigma^2$ , one or more of the  $\alpha_j$ ) that is to be investigated. A statistic for investigating  $\sigma^2$  is always included, since this statistic occurs in the tests and confidence intervals for any type of effect, and also occurs in estimates of variances for point estimates of effects.

Some of the more elementary probability properties of the statistics are stated without verification. However, proofs are easily obtained from considerations such as those given in [2]. Also, the customary results are obtained from material in [2].

Some further notation is introduced for stating the statistics that are considered for possible use.

$$\begin{aligned} y_{j.} &= \sum_{k=1}^K y_{jk}/K, & y_{.k} &= \sum_{j=1}^J y_{jk}/J \\ y_{..} &= \sum_{j=1}^J \sum_{k=1}^K y_{jk}/JK = \hat{\mu}, & \hat{\alpha}_j &= y_{j.} - y_{..} \\ s_a^2 &= \sum_{j=1}^J \hat{\alpha}_j^2/(J-1), & s_I^2 &= \sum_{j=1}^J \sum_{k=1}^K (y_{jk} - y_{j.})^2/J(K-1) \\ s_{II}^2 &= \sum_{j=1}^J \sum_{k=1}^K (y_{jk} - y_{j.} - y_{.k} + y_{..})^2/(J-1)(K-1) \\ F_{aI} &= K s_a^2/s_I^2, & F_{aII} &= K s_a^2/s_{II}^2 \end{aligned}$$

The statistic  $s_I^2$  is the customary unbiased estimate of  $\sigma^2$  for model (1) and is free of the additional random terms for extended models (4), (6), and (8). The statistic  $s_{II}^2$  is an unbiased estimate of  $\sigma^2$  and is free of the additional random terms for all the extended models. With models (4), (6), (8) and normality,  $J(K-1)s_I^2/\sigma^2$  has a  $\chi^2$ -distribution with  $J(K-1)$

degrees of freedom. For all the models and normality,  $(J - 1)(K - 1) s_{II}^2/\sigma^2$  has a  $\chi^2$ -distribution with  $(J - 1)(K - 1)$  degrees of freedom.

The statistic  $s_a^2$  is free of the additional random terms for models (2), (3), and (4). When the normality assumption also holds for the  $e_{jk}$ , and any of models (2), (3), or (4) applies, the statistic  $F_{a11}$  has an F-distribution with  $J - 1$  and  $(J - 1)(K - 1)$  degrees of freedom under the null hypothesis that the  $\alpha_j$  are all zero. This is readily verified by showing that the  $y_{j.} - y_{..}$  are uncorrelated with the  $y_{jk} - y_{j.} - y_{.k} + y_{..}$  and that, under the null hypothesis,  $K(J - 1) s_a^2/\sigma^2$  has a  $\chi^2$ -distribution with  $J - 1$  degrees of freedom and  $(J - 1)(K - 1) s_{II}^2/\sigma^2$  has a  $\chi^2$ -distribution with  $(J - 1)(K - 1)$  degrees of freedom. For normality, the  $\alpha_j$  all zero, and model (4), the statistic  $F_{a1}$  has an F-distribution with  $J - 1$  and  $J(K - 1)$  degrees of freedom (the customary result when model (1) applies).

The statistic  $\hat{\alpha}_j$  is the customary unbiased estimate of  $\alpha_j$  for model (1) and is free of the additional random terms for models (2), (3), and (4). For these extended models,  $(J - 1) s_{II}^2/JK$  is an unbiased estimate of the variance of  $\hat{\alpha}_j$  and, when the normality assumption also holds for the  $e_{jk}$ , is independent of  $\hat{\alpha}_j$  (since the  $y_{j.} - y_{..}$  are uncorrelated with the  $y_{jk} - y_{j.} - y_{.k} + y_{..}$ ). The statistic  $(J - 1) s_{II}^2/JK$  is an unbiased estimate of the variance of  $\hat{\alpha}_j$  when model (4) applies and, if the normality assumption also holds, is independent of  $\hat{\alpha}_j$  (the customary results when model (1) applies). The distribution of  $\hat{\alpha}_j$  is normal with mean  $\mu$  and variance  $(J - 1) \sigma^2/JK$  when the normality assumption applies and any of models (2), (3), or (4) holds. These properties can be used to construct  $t$ -statistics for investigating linear combinations of the  $\alpha_j$ .

Finally,  $\hat{\mu}$  is the customary unbiased estimate of  $\mu$  for model (1) and is free of the additional random terms for models (2), (5), and (6). For these extended models,  $s_{II}^2/JK$  is an unbiased estimate of the variance of  $\hat{\mu}$  and, when the normality assumption also holds for the  $e_{jk}$ , is independent of  $\hat{\mu}$  (since  $y_{..}$  is uncorrelated with the  $y_{jk} - y_{j.} - y_{.k} + y_{..}$ ). The statistic  $s_I^2/JK$  is an unbiased estimate of the variance of  $\hat{\mu}$  when model (6) holds and, if the normality assumption also applies, is independent of  $\hat{\mu}$  (the customary model (1) results). The distribution of  $\hat{\mu}$  is normal with mean  $\mu$  and variance  $\sigma^2/JK$  when the normality assumption holds and any

of models (2), (5), (6) applies. These properties can be used to construct a  $t$ -statistic for investigating  $\mu$ .

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## AN ALGORITHM FOR NONCONVEX PROGRAMMING \*

G. GRAVES and A. WHINSTON

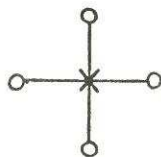
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### 1. Introduction.

This paper presents an algorithm to solve the most general mathematical programming problem

$$\begin{aligned} s.t. \quad & g^i(y) \leq 0 \quad i = 1, 2, \dots, m \\ \text{Min.} \quad & g(y) \quad y = (y_1, \dots, y_n) \end{aligned}$$

The only restriction required is that the functions  $g^i$ ,  $g$  be real valued. The general formulation allows for nonlinear or linear integer programming, mixed integer programming and general nonconvex continuous variable programming. The extant algorithms for this most general problem can usually be viewed as local search procedures. They suffer from two serious difficulties which can be characterized as the « dimensionality problem » and the problem of « trapping at local optima ». These difficulties are illustrated by the « local corner search » where each of the  $2^n$  adjacent



corners of a current point are evaluated and the best of these is used as the next current point. The number of function evaluations increases exponentially with the number of variables and the procedure is impossible except for problems with very few variables. As is well known, this procedure stabilizes at local optima. Traditionally, convexity is invoked by mathe-

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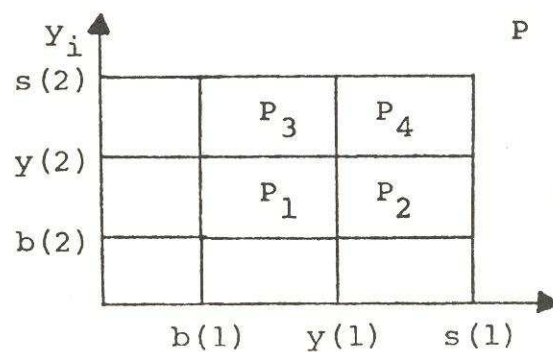
maticians to eliminate this sort of unpleasantness. As a practical matter with real problems, convexity is never established. In fact, the essence of location of facilities problems is precisely the tradeoff between the economies of scale in production and the transportation cost. (Economies of scale imply minimization of concave functions).

In a recent paper [4], Falk and Soland have presented a method which is intended for nonconvex problems where the criterion function is separable. It employs the general ideas of branch and bound where branching is effected by repeated partitioning of the domain. Bounds are obtained for the various partitions replacing the criterion function by its convex support and optimizing the resulting problem. Falk and Soland have given a proof of convergence. However, a major drawback of their method is the fact that the subproblems formed to determine the bound are in general nonconvex optimization problems. This can be seen by noting that while the subproblem's criterion function is convex by construction, the constraints are determined by intersecting the sets of points defined by the original constraint set with the constraints defining the partition. If the original constraint set is nonconvex, which is the form they support to treat, the resulting feasible region cannot be guaranteed to be convex. Presumably, the effective scope of application of their method would be limited to convex constraint sets.

The classical approaches, then, have been essentially « local » or « neighborhood » techniques dependent on derivatives (or finite difference approximations to derivatives). Only unrealistic assumptions such as « convexity » or vague arm waving such as « try a representative sample of starting points » have been advocated to deal with the global problem. (Obtaining a « representative sample of starting points » is feasible with small generally artificial examples). We feel this sweeps the very quintessence of many economic problems under the rug. Our central aim here is to present a new framework for reaching *global* optimum. The procedure involves two interconnected mechanisms, a method for structuring the search and a decision rule for selecting the course of the search.

## 2. Structuring the Search.

Structuring the search consists of introducing a framework for reducing the general problem to that of « implicit enumeration » [1] suitable for machine implementation. In general, given a bounded domain  $P$ , it can be symmetrically partitioned into components  $P_1, P_2, \dots, P_n$ . For example



Technically :

$$\text{given } b(i) \leq y(i) \leq s(i)$$

$$\text{define } r(i) = (s(i) - b(i))/2$$

$$\bar{y}(i) = b(i) + r(i)$$

and introduce the class C of finite maps

$$\omega : \{1, \dots, n\} \rightarrow \{0, 1\}$$

Now a 1 — 1 correspondence can be setup between the components  $P_i$  of the partition of  $P$  and the class of maps  $C$  by defining the upper and lower bounds of a component in terms of a map  $\omega$

$$L(i, \omega(i)) = \bar{y}(i) - (1 - \omega(i)) \cdot r(i)$$

$$U(i, \omega(i)) = \bar{y}(i) + \omega(i) \cdot r(i)$$

To illustrate these formulas, we can apply them to the two dimensional unit square. In this event,

$$0 \leq y(i) \leq 1 \quad i = 1, 2$$

$$\text{e.g. } b(1) = 0 \quad s(1) = 1$$

$$b(2) = 0 \quad s(2) = 1$$

$$\text{and } r(1) = 1/2 \quad r(2) = 1/2$$

$$\bar{y}(1) = 1/2 \quad \bar{y}(2) = 1/2$$

Using these quantities :

$$L(1, \omega(1)) = 1/2 - (1 - \omega(1)) \cdot 1/2$$

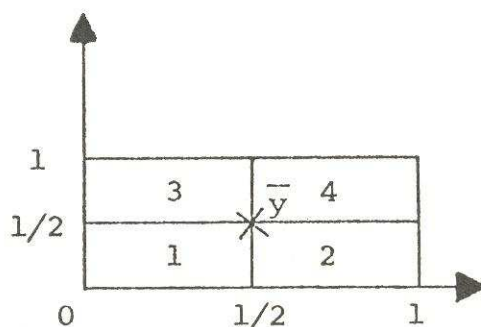
$$U(1, \omega(1)) = 1/2 + \omega(1) \cdot 1/2$$

and

$$L(2, \omega(2)) = 1/2 - (1 - \omega(2)) \cdot 1/2$$

$$U(2, \omega(2)) = 1/2 + \omega(2) \cdot 1/2$$

The choice of any of the four different maps  $(\omega(1), \omega(2))$  specifies a particular rectangle.



For example, consider the map  $(0, 1)$ , e.g.

$$\omega(1) = 0$$

$$\omega(2) = 1$$

This map specifies rectangle 3

$$L(1, \omega(1)) = 0 \quad U(1, \omega(1)) = 1/2$$

$$L(2, \omega(2)) = 1/2 \quad U(2, \omega(2)) = 1$$

The problem is now reduced to choosing a desirable map  $\omega_* \in C$  and further refining the corresponding component until a point is specified to any predetermined accuracy.

Technically this can be setup recursively by taking

$$r^0(i) = (s(i) - b(i))/2$$

$$\bar{y}^0(i) = b(i) + r^0(i)$$

and

$$r^t(i) = r^{(t-1)}(i)/2$$

$$\bar{y}^t(i) = \bar{y}^{(t-1)} - (1 - \omega^{(t-1)}(i)) \cdot r^t(i) + \omega_*^{(t-1)}(i) \cdot r^t(i)$$

and again for any map  $\omega^t$  at the  $t^{\text{th}}$  stage

$$L(i, \omega^t(i)) = \bar{y}^t(i) - (1 - \omega^t(i)) \cdot r^t(i)$$

$$U(i, \omega^t(i)) = \bar{y}^t(i) + \omega^t(i) \cdot r^t(i)$$

Now specifying a sequence of maps

$$(\omega^0, \omega^{*1}, \omega^{*2}, \dots)$$

specifies a sequence of nested intervals for each  $i$

$$[L(i, \omega^{*t}(i)) \cup U(i, \omega^{*t}(i))]$$

such that the limits of  $L(i, \omega^{*t}(i))$  monotonically increases with  $t$  and the limits of  $U(i, \omega^{*t}(i))$  monotonically decreases with  $t$  and the difference  $[U(i, \omega^{*t}(i)) - L(i, \omega^{*t}(i))] = (s(i) - b(i))/2^{t+1}$  approaches zero as  $t$  increases.

Therefore, a sequence of maps  $(\omega^0, \omega^{*1}, \dots)$  defines an  $n$ -tuple of real numbers or a point in  $R^n$ . (Recall the Weirstrauss-Heine development of the real numbers. Their definition is: « A real number is a nest of intervals  $(x_n, y_n)$  such that  $\{x_n\}$  is monotonic decreasing, and  $d_n = (y_n - x_n) \rightarrow 0$  as  $n \rightarrow \infty$  ». See Knopp, [3], Chapter 1). Now for any stipulated accuracy of the solution  $y^*(i) \pm \varepsilon$  take the first positive integer  $T$  such that  $(s^T(i) - b^T(i))/2^{T+1} \leq \varepsilon$  for all  $i$  or  $2^{T+1} \geq (s^T(i) - b^T(i))/\varepsilon$ .

For a choice of  $t/0 = T$ , introduce the class  $\bar{\omega}$  of meta-maps

$$\bar{\omega} = \{\omega^1, \omega^2, \dots, \omega^T\} \text{ or}$$

$$\Phi = \{1, 2, \dots, n \times T\} \rightarrow \{0, 1\}$$

The choice of a  $\Phi$  determines a « quantitized » point in the domain of interest. The problem is reduced to choosing the optimal meta-map  $\Phi^* = (\omega^{*1}, \omega^{*2}, \dots, \omega^{*T})$ . The algorithm we propose is to *implicitly enumerate* the class  $\bar{\omega}$  of meta-maps. There are of course many other ways of « quantitizing » the domain suitable for implicit enumeration. The employment of the present structure and, in particular, the  $T$  sub-maps  $(\omega^1, \omega^2, \dots, \omega^T)$  to specify  $\Phi$  is to isolate for easy exploitation the nested components of the successive partitions identified by the  $\omega^t$ . *It is these nested components that allow us to introduce set functionals for decision making and a global approach to calculating the optimum independent of such restrictions as convexity on the original functions.*

### 3. Decision Rules for Directing the Search.

The most common set functional in mathematics is the ordinary integral. It is our contention that use of this functional instead of resorting to the derivative or its finite difference counterpart of the « local » procedures should enable us to utilize global information. Liberating our decision

process from the myopic local neighborhood processes should render us insensitive to trapping at local optima and enable us to dispense with inapplicable mathematical assumptions such as «convexity». The most elementary use of the integral would be to simply calculate for each component (defined by an element  $\omega^t$  of the meta-map) the following quantities :

$$\begin{aligned} AV(\omega^t, g) &= \frac{1}{\prod_{i=1}^n r^t(i)} \int_{L(1, \omega^t(1))}^{U(1, \omega^t(1))} \dots \int_{L(n, \omega^t(n))}^{U(n, \omega^t(n))} g(y) dy_1 \dots dy_n \\ SS(\omega^t, g) &= \frac{1}{\prod_{i=1}^n r^t(i)} \int_{L(1, \omega^t(1))}^{U(1, \omega^t(1))} \dots \int_{L(n, \omega^t(n))}^{U(n, \omega^t(n))} g^2(y) dy_1 \dots dy_n \\ SGM(\omega, g) &= [(SS(\omega^t, g) - AV^2(\omega^t, g))]^{1/2} \\ d(\omega^t, g) &= AV(\omega^t, g) - v \cdot SGM(\omega^t, g) \end{aligned}$$

The element of the meta-map  $\omega^t$  chosen would be such that

$$d(\omega^*, g) = \min_{\omega^t \in C} d(\omega^t, g)$$

The decision functional  $d(\omega^t, g)$  is a simple estimator of the minimum value of the function  $g(y)$  on the associated component of the partition. If no knowledge of the underlying distribution is available, the parameter  $v$  in the definition of  $d(\omega^t, g)$  would have to be determined empirically or several runs made using various values.

This simple procedure suffers from the same « dimensionality problem » as the local search procedures. The evaluation of the decision functional  $d(\omega^t, g)$  for all possible  $2^n$  maps  $\omega^t$  would impose an intolerable computational burden (except for artificial mathematical examples). This « dimensionality problem » can be eliminated, however, by resorting to an  $n$ -stage sequential decision process. The total map  $\omega^t$  would be constructed in  $n$ -steps by sequentially fixing elements of the map. Suppose an arbitrary set of  $k$  out of the possible  $n$  elements of the domain are fixed. At the  $(k+1)^{st}$  step an additional element of the domain, say  $l_{k+1}$  is chosen and

$$l_{k+1} \rightarrow 0 \quad \text{or} \quad l_{k+1} \rightarrow 1$$

Now if the order of fixing elements of the domain is completely arbitrary, there would be  $2(n-k)$  possible choices of a couple  $(l_{k+1}, 0)$  or  $(l_{k+1}, 1)$  at each stage. The total number of functional evaluations would reduce to

$$\sum_{k=0}^{n-1} 2(n-k) = 2 \sum_{k=1}^n k = n(n+1)$$

(This reduction is insignificant for 3 or 4 variables, but with as few as 20 variables we would achieve a reduction from

$$2^{20} = 1,048,576$$

to

$$20 \cdot 21 = 420.)$$

In the  $n$ -stage sequential process, it is necessary to use a slightly more sophisticated decision functional. Each choice is now determined by expected values over all completions of the  $k$ -partial map. Given a  $k$ -partial map,

$$\left( \begin{array}{ccccccc} i_1, & i_2, & \dots, & i_k, & i_{k+1}, & \dots, & i_n \\ \bar{\omega}^t(i_1), & \bar{\omega}^t(i_2), & \dots, & \bar{\omega}^t(i_k), & \omega^t(i_{k+1}), & \dots, & \omega^t(i_n) \end{array} \right)$$

where the  $\{i_p\}$  are an arbitrary ordering of the elements of the domain and the barred elements are considered fixed, we need to consider its completion class  $C_k^t$ . This completion class consists of all possible completions leaving the first  $k$  assignments fixed. It is the subset of maps which give the same assignment for a specific set of  $k$  elements of the domain. We then employ the following expected values over the completion class  $C_k^t$

$$\begin{aligned} E_{C_k^t}(AV(\omega^t, g)) &= \frac{1}{2^{n-k} \prod r(j)} \int_{L(1, \bar{\omega}^t(1))}^{U(1, \bar{\omega}^t(1))} \dots \int_{L(k, \bar{\omega}^t(k))}^{U(k, \bar{\omega}^t(k))} \\ &\quad \dots \int_{L(k+1, 0)}^{U(k+1, 1)} \dots \int_{L(n, 0)}^{U(n, 1)} g(y) dy_1 \dots dy_n \\ E_{C_k^t}(SS(\omega^t, g)) &= \frac{1}{2^{n-k} \prod r(j)} \int_{L(1, \bar{\omega}^t(1))}^{U(1, \bar{\omega}^t(1))} \dots \int_{L(k, \bar{\omega}^t(k))}^{U(k, \bar{\omega}^t(k))} \\ &\quad \dots \int_{L(k+1, 0)}^{U(k+1, 1)} \dots \int_{L(n, 0)}^{U(n, 1)} g^2(y) dy_1 \dots dy_n \end{aligned}$$

These results, of course, rely on the « additivity » of the limits of integration.

Using these more sophisticated quantities we proceed as before by calculating

$$SGM^{(k)}(\omega^t, g) = [E_{C_k^t}(SS(\omega^t, g)) - E_{C_k^t}^2(AV(\omega^t, g))]^{1/2}$$

that is, the standard deviation of  $g(y)$  on the components and

$$d^{(k)}(\omega^t, g) = E_{C_k^t}(AV(\omega^t, g)) - v \cdot SGM^{(k)}(\omega^t, g)$$

The decision functional  $d^{(k)}(\omega^t, g)$  is evaluated for the  $2(n-k)$  possible couples, say  $(l_{k+1}) \rightarrow 1$  or  $(l_{k+1}, 0)$  and  $l_{k+1}$  any « free » element of the  $k$ -partial map. The minimum value of  $d^{(k)}(\omega^t, g)$  determines the next couple to be fixed.

This whole  $n$ -stage sequential decision process is then carried out  $T$  times as indicated in Section 1 to yield a « point » in  $R^n$  which is hopefully very close to the global minimum of  $g(y)$ . In any event, by continuing and employing a « confidence level implicit enumeration » (see [1] and [2]) of the whole class  $\bar{c}$  of meta-maps, we should achieve a highly sophisticated search of the whole domain. The only point to note in employing the mechanism of the « confidence level enumeration » is that the recursive definition of the components would require  $\omega^i$  to be entirely fixed before any element of  $\omega^{i+1}$ .

#### 4. Additional Observations.

##### (A) Limiting value.

When the function  $g(y)$  is continuous, it might be worth noting that

$$d^{(n)}(\omega^t, g) \rightarrow g(y^*) \quad \text{as } t \rightarrow \infty$$

$$\text{where } y^* \in R^n$$

is the point defined by the sequence of maps  $(\omega^0, \omega^{*1}, \dots)$ . This follows immediately from the Mean Value Theorem for Integrals which says :

$$\frac{1}{\text{AREA}} \int_D g dA = g(\tilde{y})$$

where  $\tilde{y} \in D$ .

Applying this result to the terms of  $d^{(n)}(\omega^t, g)$  yields

$$E_{C_n}(AV(\omega^t, g)) \rightarrow g(y^*) \quad \text{as } t \rightarrow \infty$$

$$SGM^{(n)}((\omega^t, g)) \rightarrow 0 \quad \text{as } t \rightarrow \infty$$

and hence

$$d^{(n)}(\omega^t, g) \rightarrow g(y^*) \quad \text{for any } v \text{ as } t \rightarrow \infty.$$

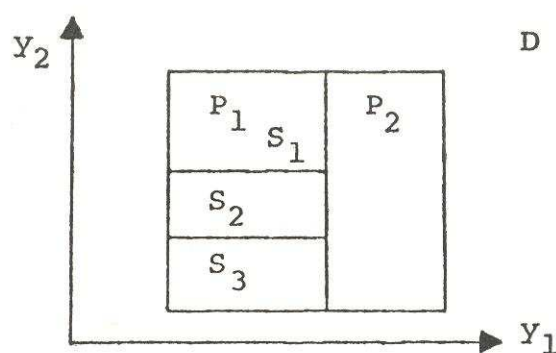
##### (B) Indefinite Integral.

The evaluation of the integrals employed in the definition of the decision functional  $d^{(k)}(\omega^t, g)$  can be carried out in various ways. With continuous functions, the simplest procedure is to employ the closed form given by the indefinite integral; for example,

$$\begin{aligned} \frac{1}{\text{AREA}} \int_D y_1 y_2 dy_1 dy_2 &= \frac{(U_1^2 - L_1^2) \cdot (U_2^2 - L_2^2)}{4(U_1 - L_1) \cdot (U_2 - L_2)} \\ &= \frac{(U_1 + L_1) \cdot (U_2 + L_2)}{4} \end{aligned}$$

## (C) Stratified Sampling.

When the function is not known in closed form or the indefinite integral is not available, it may become necessary to resort to stratified sampling of the various components of the domain defined by the limits of integration in the decision functional; for example,



We could determine an appropriate sample size  $k_i$  for each strata  $S_i$  and on the basis of this sample calculate estimates

$$\hat{E}_{c_k} (AV(\omega^t, g)) \quad \text{and} \quad \hat{SGM}^{(k)}(\omega^t, g)$$

and from these calculate  $d^{(k)}(\omega^t, g)$ . At any decision point, we are stratifying a domain of the form :

$$\begin{aligned} L(1, \omega^t(1)) &\leq y_1 \leq U(1, \omega^t(1)) \\ &\vdots \\ L(k, \omega^t(k)) &\leq y_k \leq U(k, \omega^t(k)) \\ L(k+1, 0) &\leq y_{k+1} \leq U(k+1, 1) \\ &\vdots \\ L(n, 0) &\leq y_n \leq U(n, 1) \end{aligned}$$

(It is also possible to apply Analysis of Variance and other more sophisticated statistical techniques in refining the components).

## (D) Discrete Variables.

It is, of course, not necessary that the variables be continuous. The Riemann-Stieltjes Integral is available to deal with discrete variables. Recall the usual Unit Step Function

$$I(y) = \begin{cases} 0 & (y \leq 0) \\ 1 & (y > 0) \end{cases}$$

and Standard Counting Measure

$$\alpha_i(y_i) = I(y_i) + I(y_i - 1)$$

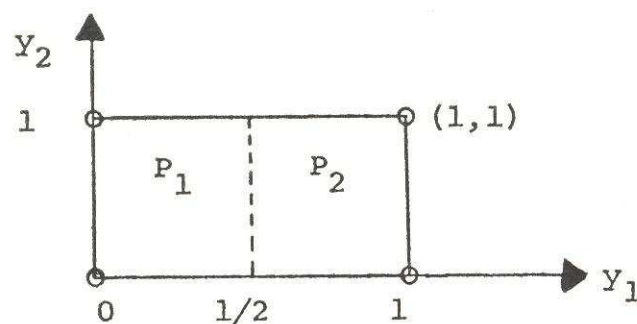
that would be employed with zero-one discrete variables. (A slight generalization would eliminate the reduction to zero-one discrete variables). In this formulation,

$$\llcorner \text{AREA} \gg = \int_0^{1+\epsilon} \dots \int_0^{1+\epsilon} d\alpha_1 \dots d\alpha_n = 2^n$$

and for illustrative purposes, consider the simple linear case

$$g(y) = \sum_{i=1}^n a_i y_i$$

Take  $S(i) = 1 + \epsilon$  and  $b(i) = 0$



$$\begin{aligned} P_1 \quad E_{C_k}(AV(\omega^t, g)) &= \frac{1}{2} \int_0^{1+\epsilon} \int_0^{1/2} (a_1 y_1 + a_2 y_2) d\alpha_1 d\alpha_2 \\ &= \frac{1}{2} \int_0^{1+\epsilon} a_2 y_2 d\alpha_2 = a_2/2 \end{aligned}$$

$$\begin{aligned} P_2 \quad E_{C_k}(AV(\omega^t, g)) &= \frac{1}{2} \int_0^{1+\epsilon} \int_{1/2}^{1+\epsilon} (a_1 y_1 + a_2 y_2) d\alpha_1 d\alpha_2 \\ &= \frac{1}{2} \int_0^{1+\epsilon} (a_1 + a_2 y_2) d\alpha_2 = a_1 + a_2/2 \end{aligned}$$

Hence, as expected, the decision of whether  $y_1 \rightarrow 0$  or  $y_1 \rightarrow 1$  is determined by whether  $a_1 < 0$  or  $a_1 > 0$ . This general approach reduces to techniques expounded in great detail in the paper, «A New Approach to Discrete Mathematical Programming». It should be stressed that the Riemann-Stieltjes Integral Approach developed in this section is perfectly capable of handling

pure continuous variables, mixed continuous and integer variables, or pure integer variables.

(E) Constraints.

The ideas developed in this paper can be extended to treat constraints of the form

$$g_1(y) \leq 0$$

by introducing conditional expected values. The simplest way to achieve this is through the use of a Regression Equation. Instead of using  $AV(\omega^t, g)$ , this would require employment of :

$$AV(\omega^t, (g_1|g_2)) = AV(\omega^t, g_2) + \frac{COV(\omega^t, g_1, g_2)}{Var(g_1)} \cdot (g_1 - AV(\omega^t, g_1))$$

the conditional expected value of the function  $g_2$  given a value of function  $g_1$ . In this procedure, it would be necessary to estimate the maximum or minimum of  $(g_1 - AV(\omega^t, g_1))$  depending on the sign of the covariance on the components specified by the current  $k$ -partial map  $\omega^t$ . This could be done in turn in terms of the variance of  $g_1$  and its mean. It would also be necessary to establish an appropriate confidence level that  $g_1(y) \leq 0$  on the component. When the confidence drops to low it is necessary to « backtrack » in the construction of the meta-map. It should be observed that « normality » assumptions are not required for this procedure, but in the event of non-normality, the linear regression equation reduces to a first-order approximation. Again, these ideas are developed at greater length in [1].

## 5. Examples.

The following simple examples will illustrate some of the ideas developed in this paper.

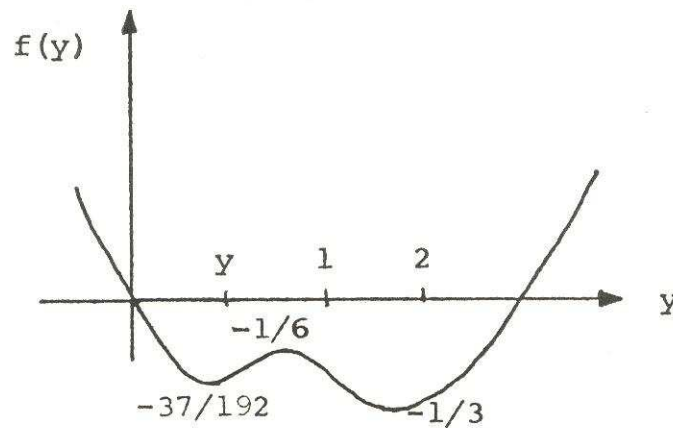
*Example 1* (single variable two minima).

$$g(y) = 1/4 y^4 - 7/6 y^3 + 7/4 y^2 - y$$

$$y = 1/2 \quad f(y) = -37/192 \quad (\text{local minimum})$$

$$y = 1 \quad f(y) = -1/6 \quad (\text{local maximum})$$

$$y = 2 \quad f(y) = -1/3 \quad (\text{global minimum})$$



*Subroutine.*

*Sum.*

$$S = 1/20 (U^5 - L^5) + 7/24 (U^4 - L^4) + 7/12 (U^3 - L^3) - 1/2 (U^2 - L^2)$$

*Sum of Squares.*

$$\begin{aligned} SS = & 1/144 (U^9 - L^9) - 7/96 (U^8 - L^8) + 161/504 (U^7 - L^7) \\ & - 55/72 (U^6 - L^6) + 259/240 (U^5 - L^5) - 7/8 (U^4 - L^4) \\ & + 1/3 (U^3 - L^3) \end{aligned}$$

*Results.*

$$\begin{aligned} \text{Run 1} \quad b(1) &= 0 & s(1) &= 3 & t &= 1.0 \\ y^{(0)} &= 1.5 \end{aligned}$$

Where linear damping  $v^1 = -\left(\frac{Mr - IT}{Mr}\right) v$  was employed and,  $Mr$  is

the total number of steps and  $IT$  is the current step.

*Example 2 (two variables).*

$$f(y) = 3(y_1 - 2)^2 + (y_2 - 2.5)^2$$

*Sum.*

$$\begin{aligned} S = & (U_2 - L_2)(U_1^3 - L_1^3) - 6(U_1^2 - L_1^2) \\ & + (U_1 - L_1)(1/3(U_2^3 - L_2^3) - 5/2(U_2^2 - L_2^2) + 18.25(U_2 - L_2)) \end{aligned}$$

|                 | AV      | SGM                | $d$      |
|-----------------|---------|--------------------|----------|
| $\omega(1) = 0$ | — .1687 | .0412              | — .2065  |
| $\omega(1) = 1$ | .0187   | .4816              | — .3859  |
|                 |         | $y^{(1)} = 2.25$   |          |
| $\omega(1) = 0$ | — .3005 | .0287              | — .3208  |
| $\omega(1) = 1$ | .3380   | .5090              | .0410    |
|                 |         | $y^{(2)} = 1.8750$ |          |
| $\omega(1) = 0$ | — .2817 | .0264              | — .2938  |
| $\omega(1) = 1$ | — .3194 | .0165              | — .3248  |
|                 |         | $y^{(3)} = 2.0625$ |          |
| $\vdots$        |         | $\vdots$           | $\vdots$ |
| $\omega(1) = 0$ | — .3333 | .0284              | — .3333  |
| $\omega(1) = 1$ | — .3332 | .0191              | — .3332  |
|                 |         | $y^{(8)} = 1.998$  |          |

Runs using the straight mean were made on the following domains :

$$\begin{array}{ll}
 b(1) = -1 & s(1) = 1 \\
 b(2) = -5 & s(2) = 5 \\
 b(1) = -2 & s(1) = 2 \\
 b(2) = -1 & s(2) = 1 \\
 b(1) = -5 & s(1) = 5 \\
 b(2) = -5 & s(2) = 5
 \end{array}$$

In all cases the minimum was attained.

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Further information concerning the Symposium may be obtained from Professor Richard W. Cottle at the same address.

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