Dice representability of reciprocal relations

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We define a dice model $\mathcal{D}$ as a tuple $(D_1, \ldots, D_m, Q)$ consisting of $m \geq 2$ dice, where each dice D_i is represented by the multiset of integers present on its n_i faces, and a rational-valued reciprocal relation Q expressing the winning probability of one dice over another, henceforth called the winning probability relation of $\mathcal{D}$, defined by $Q(D_i, D_j) = \text{Prob}\{D_i > D_j\} + \frac{1}{2}\text{Prob}\{D_i = D_j\}$ where

$$\text{Prob}\{D_i > D_j\} = \#\{\, (x, y) \in D_i \times D_j \mid x > y \,\}/(n_i n_j)\,,$$
$$\text{Prob}\{D_i = D_j\} = \#\{\, (x, y) \in D_i \times D_j \mid x = y \,\}/(n_i n_j)\,.$$

The transitivity of winning probability relations has been studied extensively by some of the present authors, resulting in a new framework for studying transitivity of reciprocal relations in general, referred to as the cycle transitivity framework [1, 2]. It is known that the winning probability relation of a dice model, which amounts to the pairwise comparison of a set of independent random variables that are uniformly distributed on finite integer multisets, is dice transitive [4].

We can represent the winning probability relation Q of some dice model $\mathcal{D}$ as a so-called winning probability graph, where the vertices correspond to the dice and the edges (D_i, D_j) have a weight $Q(D_i, D_j)$. The condition of dice transitivity, also called the 3-cycle condition as it imposes a constraint on the weight of edges in cycles of length 3 in the winning probability graph, is not sufficient for an arbitrary rational-valued reciprocal relation to be the winning probability relation of a dice model. We therefore introduce an additional necessary condition that is a stronger version of the so-called Ferrers property. A reciprocal relation Q fulfills the 4-cycle condition if for the consecutive weights (t_1, t_2, t_3, t_4) in any 4-cycle in its winning probability graph it holds that

$$t_1 + t_2 + t_3 + t_4 - 1 \geq t_1 t_3 + t_2 t_4 + \min(t_1, t_3)\min(t_2, t_4)\,.$$

We have proven that a given rational-weighted 4-cycle and reciprocally weighted inverse cycle, both fulfilling the 4-cycle condition, can be extended to a winning probability graph representing the winning probability relation relation of some dice model consisting of four dice [3].

Références

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[4] B. De Schuymer, H. De Meyer, B. De Baets, and S. Jenei. On the cycle-transitivity of the dice model. *Theory and Decision*, 54 (2003) 261–285.