(Bio-)Mechanical Engineering Applications of Convex Optimization

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Purpose of this Talk

- a very brief introduction to convex optimization
  - nonlinear programs for which every local optimum is also globally optimal
  - fundamentally more tractable than general nonlinear programs

- illustrate practical use with two recent (bio)mechanical applications:
  - counterweight balancing
  - dynamic musculoskeletal analysis

Outline

- a convex optimization primer
- application 1 – counterweight balancing
- application 2 – dynamic musculoskeletal analysis
- conclusions and outlook

A Convex Optimization Primer

In fact the great watershed in optimization isn't between linearity and nonlinearity, but convexity and nonconvexity – *R.T. Rockafellar*

- convex programs are a specific class of nonlinear optimization problems:
  - all local optima are global
  - very efficient algorithms to solve (= find global optimum)
  - can be considered as nonlinear extension of linear programming

- fundamentally more tractable – some good news and some bad news
  - surprisingly many engineering problems can be solved via convex optimization
  - but, convex optimization problems are often difficult to recognize
    - linear programming: a few standard tricks (one norm, infinity norm)
    - convex programming: many tricks, not obvious which ones to use

Mathematical Definition

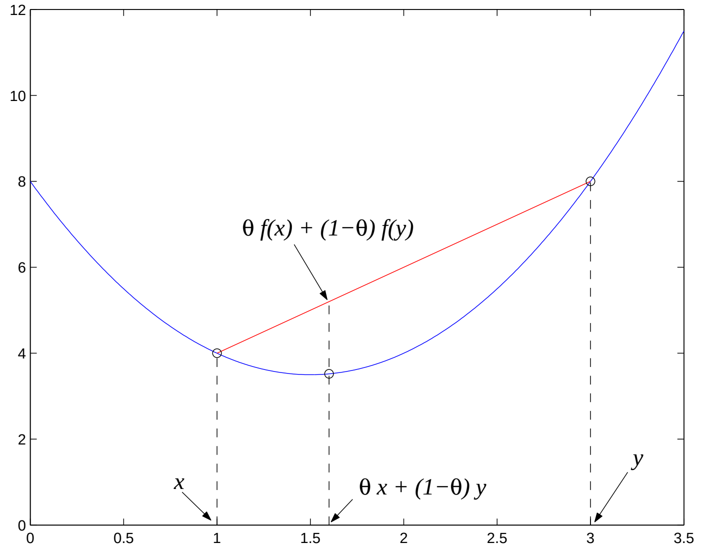
$$\begin{array}{ll} \text{minimize} & f_0(x) \\ \text{subject to} & f_i(x) \leq 0, \quad i = 1, \ldots, m \\ & a_i^T x = b, \quad\;\; i = 1, \ldots, p \end{array}$$

- objective function f_0 must be convex
- inequality constraint functions $f_1, \ldots, f_m$ must be convex
- equality constraints must be affine

Convex Function

1. $\mathbf{dom} f$ is a convex set (that is, an interval for a scalar function)

2. $\forall x, y \in \mathbf{dom} f, \theta \in [0,1]: \; f(\theta x + (1-\theta)y) \leq \theta f(x) + (1-\theta) f(y)$



Convexity Proofs

- How to prove that an optimization problem is convex?
  - proof convexity of goal function
  - proof convexity of constraint functions
- How to prove that a function is convex?
  - 'proper' combination of known, convex functions
  - second derivative is nonnegative
  - . . .
- What if the optimization problem is not convex?
  - reformulate by using a different parametrization
  - reformulate by applying superposition principles

  $\Rightarrow$ focus on formulating, not on solving optimization problems (solution = 'trivial')

- S. Boyd, L. Vandenberghe, *Convex Optimization*, Cambridge University Press, 2004

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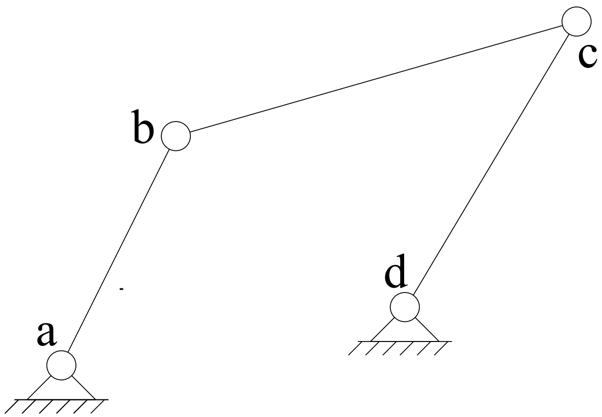
Mechanical Mechanisms

- (mechanical) mechanisms
  - mechanical devices to convert input motion/force into some output motion/force
  - for instance, crank-rocker four-bar mechanism: rotary $\longrightarrow$ oscillating (very popular)

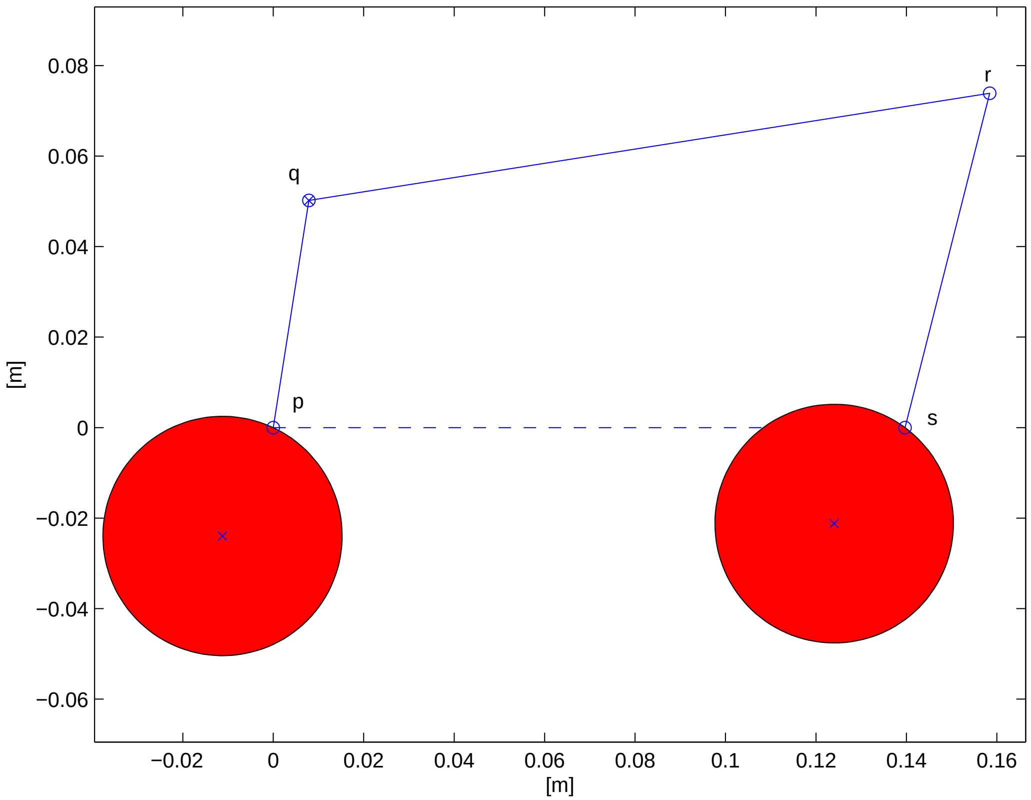


Counterweight Balancing

- to add counterweights to reduce vibration

- rotating machinery (rotors)
  - compensate for unbalance
  - e.g. car wheel and tire, turbines

- reciprocating machinery (mechanisms)
  - reduce forces/moments on supporting frame: *shaking force* and *shaking moment*
  - e.g. crankshaft counterweights (internal combustion engine), crank-rocker four-bar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Our Contribution

- counterweight balancing: nonlinear optimization problem

- our contribution:
  - theory: nonlinear change of variables $\Rightarrow$ convex problem
  - theory: methodology for mechanisms of arbitrary complexity
  - practice: trade-off charts, with ultimate balancing limits

(One Variant of) Original Nonlinear Problem

- optimization variables: cylindrical counterweight parameters: $(R_i^*, X_i^*, Y_i^*, t_i^*)$
- goal function: relative RMS of shaking moment: $\alpha_{msh} = \frac{\mathsf{rms}(M_{shak,optim})}{\mathsf{rms}(M_{shak,orig})}$
- upper limit on relative RMS of shaking force and driving torque:

$$\alpha_{fsh} \leq \alpha_{fsh}^{\mathsf{M}} \quad ; \quad \alpha_{drv} \leq \alpha_{drv}^{\mathsf{M}}$$

- upper limit on total added mass: $\sum m_i^* \leq m_{tot}^{\mathsf{M}}$.
- mass constraints for every link i:
  - positive radius and thickness: $R_i^* \geq 0 \quad ; \quad t_i^* \geq 0$
  - bounded COG coordinates: $X_i^{\mathsf{m}} \leq X_i^* \leq X_i^{\mathsf{M}} \quad ; \quad Y_i^{\mathsf{m}} \leq Y_i^* \leq Y_i^{\mathsf{M}}$

Nonlinear Change of Variables (Moments of Mass Distribution)

- $(R_i^*, X_i^*, Y_i^*, t_i^*) \Rightarrow (m_i^*, X_i^*, Y_i^*, J_i^*) \Rightarrow (\mu_{1i}^*, \mu_{2i}^*, \mu_{3i}^*, \mu_{4i}^*)$

$$\left\{ \begin{array}{rcl} \mu_{1i}^* &=& m_i^* \quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad = \quad \pi \cdot (R_i^*)^2 \cdot t_i^* \cdot \rho; \\ \mu_{2i}^* &=& m_i^* \cdot X_i^*; \\ \mu_{3i}^* &=& m_i^* \cdot Y_i^*; \\ \mu_{4i}^* &=& J_i^* + m_i^* \cdot \big((X_i^*)^2 + (Y_i^*)^2\big) \quad = \quad m_i^* \cdot \left(\frac{(R_i^*)^2}{2} + (X_i^*)^2 + (Y_i^*)^2 \right). \end{array} \right.$$

- Key element #1: mass constraints are convex in μ^*
  - 2D: affine and SOC $\Rightarrow$ SOCP
  - 3D: affine and LMI $\Rightarrow$ SDP
- Key element #2: all forces are linear in $\mu \Rightarrow \alpha^2$ convex quadratic in μ
  - general method developed for deriving these expressions numerically
- Key element #3: superposition: $\mu = \mu^\circ + \mu^* \Rightarrow \alpha^2$ convex quadratic in μ^*

Numerical Results – Trade-Off Curves

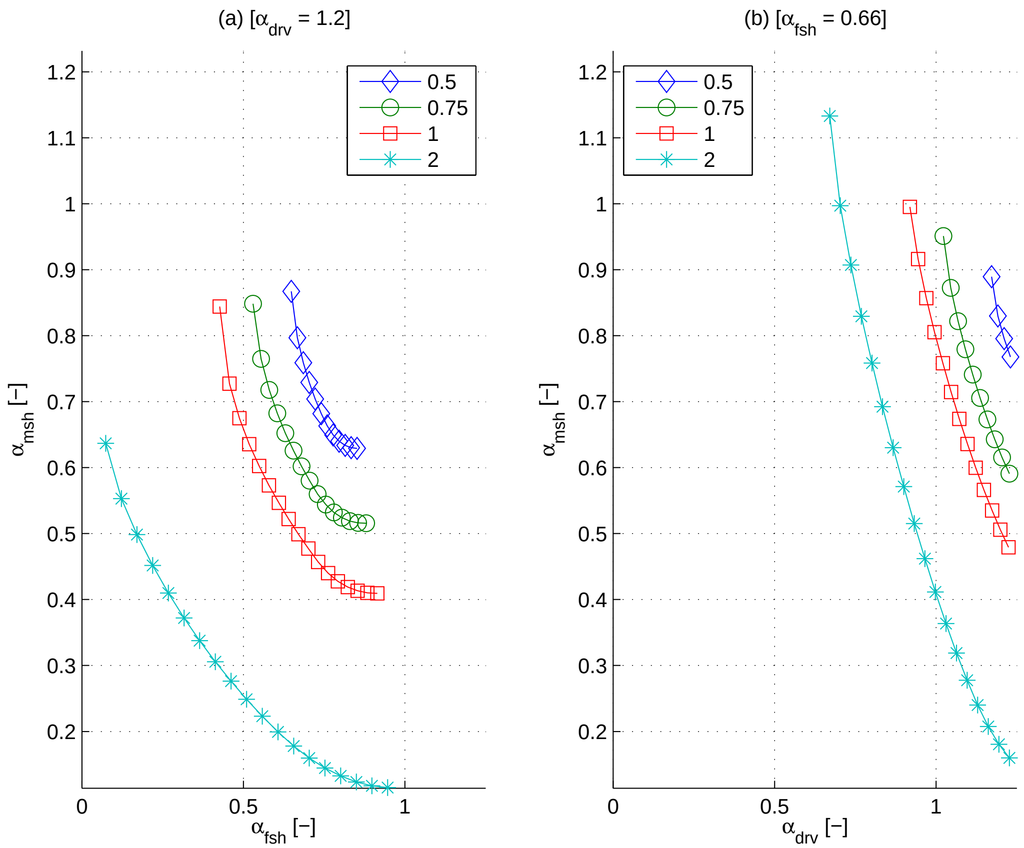
minimize α_{msh}

subject to $\alpha_{fsh} \leq \alpha_{fsh}^{\mathsf{M}}$

$\alpha_{drv} \leq \alpha_{drv}^{\mathsf{M}}$

$\sum m_i^* \leq m_{tot}^{\mathsf{M}}$

point mass constraints



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Human Motion Analysis

- goal: determine muscle forces that give rise to given human motion pattern

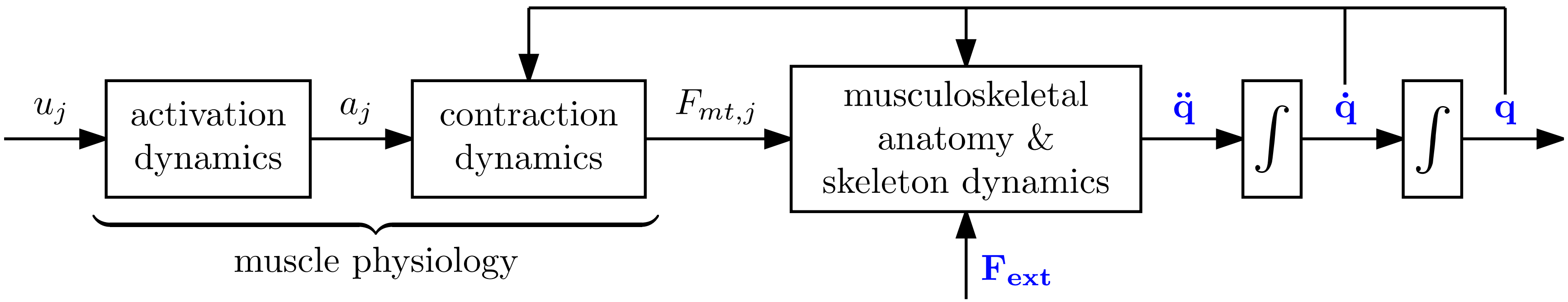
- applications include:
  - treatment of gait pathology
  - joint prosthesis
  - ergonomics and sports

- major complications:
  - muscle forces not experimentally measurable $\Rightarrow$ need to be simulated
  - human body mechanically overactuated (more muscles than degrees of freedom)

  $\Rightarrow$ given motion, skeleton's equations of motion do not yield unique solution

  $\Rightarrow$ muscle force determination reformulated as optimization problem

Modeling for Human Motion Analysis



u_j	muscle excitation
a_j	muscle activation
$F_{mt,j}$	musculotendon force
$\mathbf{F_{ext}}$	external forces
$\mathbf{q}$	generalized coordinates
j	muscle index; $j : 1, \ldots, J$
blue	experimentally measurable

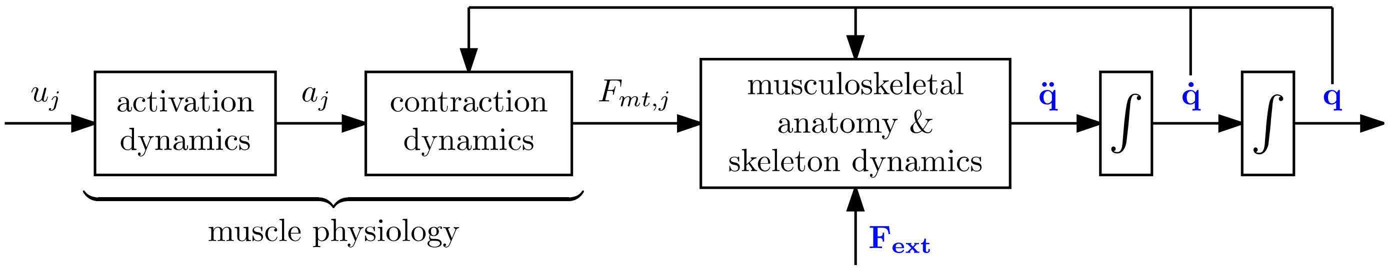
Optimization Approaches

- classical forward approach (not further discussed here)
  - computationally very expensive
  - muscle physiology included

- classical inverse approach
  - computationally inexpensive: per time instant one small-scale LP or QP
  - muscle physiology not included $\Rightarrow$ only slow motions

- physiological inverse approach (our contribution)
  - inverse approach combined with muscle physiology
  - computationally very expensive, at first sight: large-scale, nonconvex program
  - kept tractable, however, using convex optimization techniques

Physiological Inverse Approach – Original Optimization Problem

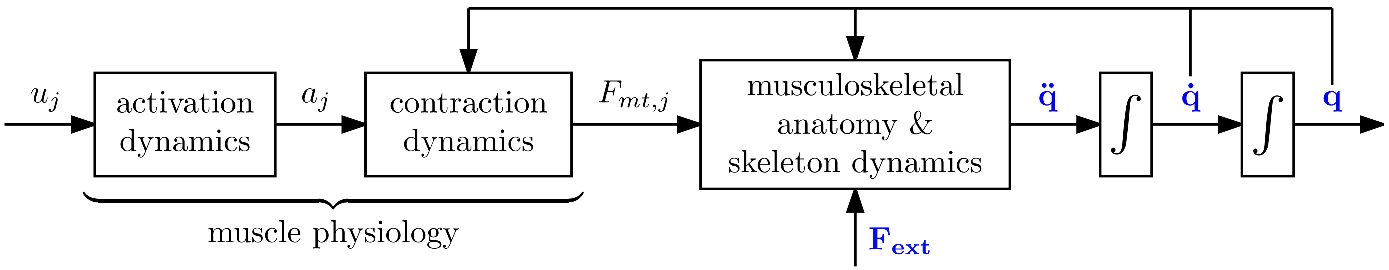


- variables: $u_j(t_k), a_j(t_k), F_{mt,j}(t_k)$ $\quad (j = 1, \ldots, J;\ k = 1, \ldots, K)$

- goal function: muscle fatigue, modeled (?) by convex function:

$$\sum_{k=1}^{K} \left[\sum_{j=1}^{J} \left(\frac{F_{mt,j}(t_k)}{C_j} \right)^2 \right]$$

Physiological Inverse Approach – Original Optimization Problem



- constraints
  - skeleton dynamics: at each t_k: linear equation in $F_{mt,j}(t_k)$, given $(q_j, \dot{q}_j, \ddot{q}_j, F_{ext})$
    $\Rightarrow$ set of (sparse) linear equality constraints
  - excitation dynamics: for each muscle j: $\frac{da_j}{dt} = f_j(u_j, a_j)$
    $\Rightarrow$ set of (sparse) nonlinear equality constraints
  - contraction dynamics: for each muscle j: $\frac{dF_{mt,j}}{dt} = f_j(a_j, F_{mt,j})$
    $\Rightarrow$ set of (sparse) nonlinear equality constraints

Physiological Inverse Approach – Our Contribution

- original optimization problem
  - large-scale, sparse, nonconvex: nonlinear equality constraints (muscle physiology)
  - nonlinear change of variables to convert to convex program: not found yet

- alternative approach: initialization for local optimization
  - step 1: find approximate convex program
  - step 2: use global optimum of approximate CP as hot-start for original program

- approximate convex program for this case?
  - keep optimization variables
  - keep goal function: already convex
  - skeleton dynamics: already convex: linear equality constraints
  - muscle physiology: turn into linear equality constraints by global linearization
    $\Rightarrow$ so far: (heuristic, but good enough) system identification techniques

Case Study: Gait Analysis

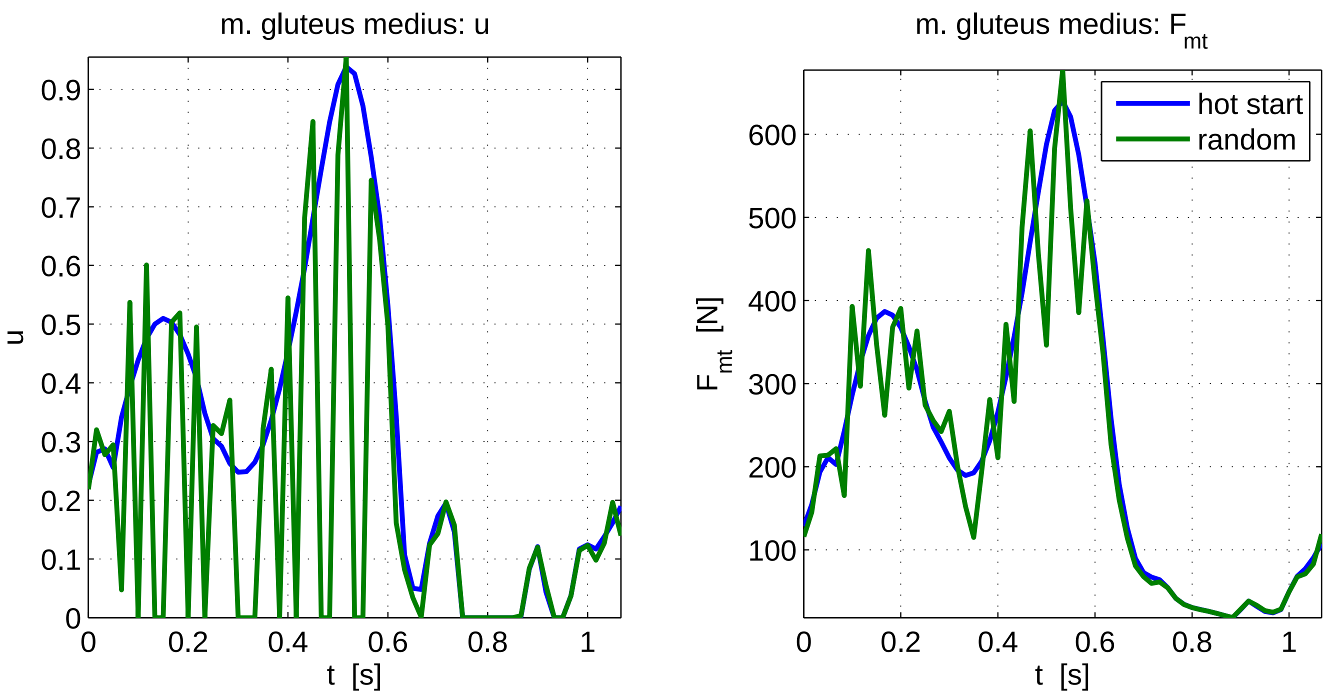
- input data
  - experimentally measured $\mathbf{q}$, $\dot{\mathbf{q}}$, $\ddot{\mathbf{q}}$, $\mathbf{F}_{\mathbf{ext}}$ (inverse simulation)
  - musculoskeletal model

- physiological inverse approach

	original nonconvex	approximate convex
# variables	7605	7605
# constraints: lin.	2600	5135
nonlin.	2535	0
solver	`snopt`	`qp-minos`

- original, nonconvex program: one run from hot-start
  - 144 CPU sec (one time nonconvex from hot-start + approximate convex)
  - nonconvex optimum: well-behaved

- original, nonconvex program, 100 runs from random starting point (55 converged)
  - 128 CPU hours (100 times nonconvex from random)
  - nonconvex optimum: ill behaved
  - goal function value: 2% to 14% worse



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Conclusions and Outlook – Counterweight Balancing

- counterweight balancing: nonlinear optimization problem
- our contribution:
  - theory: nonlinear change of variables $\Rightarrow$ convex problem
  - theory: methodology for mechanisms of arbitrary complexity
  - practice: trade-off charts, with ultimate balancing limits $\Rightarrow$ cf. Carnot
- recent developments
  - direct minimization of vibration energy of supporting frame: convex (under not too strict decoupling assumptions)
  - interesting link with pdf's (Chebyshev bounds)
  - direct optimization of link shape in 2D and 3D (large QPs/QCQPs)
  - incorporation of elastic stress considerations

Conclusions and Outlook – Dynamic Musculoskeletal Analysis

- physiological inverse approach: combination of inverse approach with muscle physiology.

- corresponding large-scale, nonlinear optimization problem can be solved very efficiently
  - initialization for local optimization
  - approximate convex program obtained by global linearization

- the first numerical results indicate a lot of potential
  - from a numerical point of view
  - from a physiological point of view
    * superficial muscles: excitation patterns correspond qualitatively to EMG
    * deep muscles (no EMG available): essential muscle actions during gait predicted
    * good correspondence between convex and nonconvex results

Conclusions and Outlook – Dynamic Musculoskeletal Analysis

- future work:
  - testing for fast motions: muscle physiology even more important
  - further exploitation of problem structure to increase numerical efficiency
  - measures to tackle sensitivity for kinematic measurement errors

- recent developments:
  - nonlinear change of variables for one particular excitation model $\Rightarrow$ convex program
  - confirms that hot-start + nonconvex program yields same global optimum

Developing a working knowledge of convex optimization can be mathematically demanding, especially for the reader interested primarily in applications. In our experience (mostly with graduate students in electrical engineering and computer science), the investment often pays off well, and sometimes very well.

Stephen Boyd and Lieven Vandenberghe in
Convex Optimization

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